Arbitrary Action Model Logic and Action Model Synthesis

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***Abstract*—We present a method for synthesising action models that result in a given post-condition when executed on any Kripke model. Action models represent social actions that affect the knowledge or beliefs of agents in multi-agent systems. In the consideration of action model synthesis, we introduce an extension of the action model logic of Baltag, Moss and Solecki [4] with an action model quantifier, $\exists\phi$ which stands for "there is an action model that results in the post-condition ϕ". We show that this quantifier is equivalent to the refinement quantifier of van Ditmarsch and French [10], and provide a sound and complete axiomatisation for the resulting logic, along with decidability and expressivity results.**

I. Introduction

In multi-agent systems, informative updates are events that lead to agents receiving new information about the world, whilst leaving the facts of the world itself unchanged. Action models, as introduced by Baltag, Moss and Solecki [4] are a model for informative updates in multi-agent systems, used to compute and reason about the changes in knowledge or belief that result from performing informative updates. We are interested in the problem of synthesising action models that will change the knowledge or beliefs of agents in such a way that their resulting knowledge or beliefs will satisfy a given post-condition when the corresponding informative update is performed.

A public announcement is a simple example of an informative update, where a statement is announced to every agent at once, usually resulting in the statement becoming common knowledge to the agents. We can imagine more complex examples of informative updates, where information is not necessarily communicated publicly, and where each agent potentially receives different information. For example: private announcements, where some agents receive information, and the other agents are unsure of what information was communicated; completely private (or secret) announcements, where the other agents are not even aware that communication took place; or deceptive informative updates, where the other agents may be mislead to believe that different information was communicated than was actually the case. From the point of view of the agents, these kinds of informative updates may involve a degree of uncertainty as to exactly what informative update is taking place. For example, a player in a game of poker may privately look at their hand of cards and learn which cards they are holding. An onlooker will know from watching that the player has learned which cards they are holding, but without learning exactly which cards the player is holding the onlooker will be uncertain as to exactly what the player has learned as a result of looking at their cards. From the onlooker's perspective there is the possibility that one of several informative updates has actually occurred, each informative update corresponding to a possible hand of cards that the player could have been dealt.

Action models generalise the kinds of informative updates we have mentioned so far. In epistemic logics, the knowledge of a collection of agents is represented by a relational structure known as a Kripke model, where relationships between "possible worlds" represent an agent's uncertainty about which possible world is the real one. Action models are similar relational structures, where relationships between "action points" represent an agent's uncertainty as to which action actually occurred; thus it is possible for an action model to comprise multiple actions, allowing for informative updates where each agent potentially receives different information, where agents may have uncertainty about which action really occurred, and where information may be communicated publicly, privately or otherwise. An action model (representing an informative update) may be executed on a Kripke model (representing an initial state of knowledge) to yield a new Kripke model (representing the state of knowledge resulting from performing the informative update from the initial knowledge state). The paper by Baltag and Moss [3] considers numerous examples of informative updates and their representations as action models.

The action model logic introduced by Baltag, Moss and Solecki [4] extends modal epistemic logics with an operator for reasoning about the effects of a specific informative update, represented by an action model. The statement $[\alpha]\phi$ stands for "ϕ is true after performing the informative update α". Baltag, Moss and Solecki [4] provide a sound and complete axiomatisation for the action model logic and show that it is decidable and expressively equivalent to its underlying modal logic. Baltag and Moss [3] show that the action model logic is a generalisation of other logics that introduce similar operators for more restricted varieties of informative updates, such as the logics of public announcements of Plaza [18] and Gerbrandy and Groenvald [13], the logic of completely private announcements to groups of agents of Gerbrandy and Groenvald [13], and the logic of common knowledge of alternatives of Baltag, Moss and Solecki [4] and van Ditmarsch [9].

In the present work, we are interested in the problem of action model synthesis. We aim, for a given post-condition ϕ, to find an action model α that will result in ϕ becoming true when it is executed on any Kripke model. Of course it is often the case that whether there is any action model that can result in a given post-condition will depend on the particular Kripke model that we are interested in executing it on. We therefore concurrently try to answer the question of whether for any particular Kripke model there exists an action model that will result in a given post-condition. To these ends we introduce the arbitrary action model logic, extending the action model logic of Baltag, Moss and Solecki [4] with an action model quantifier, $\exists\phi$ which stands for "There is an action model that results in the post-condition ϕ ". Action model quantifiers allow us to pose questions about which post-conditions can and cannot be achieved through the execution of arbitrary action models. We use action model quantifiers in our action model synthesis results, to produce action models that will result in a given post-condition in any Kripke model where the post-condition can be achieved through an action model.

The question of quantifying over informative updates such as action models has previously been considered in settings with different models for informative updates. When the informative updates we consider are restricted to public announcements, we get the arbitrary public announcement logic, introduced by Balbiani, et al. [2]. The arbitrary action model logic was suggested by Balbiani, et al. as a possible generalisation of the arbitrary action model logic. A similar logic, the group announcement logic of Ågotnes, et al. [1], quantifies over public announcements that specifically consist of a conjunction of statements that are each known by at least one agent in a given group of agents. Yet another similar logic, the refinement modal logic of van Ditmarsch and French [10] quantifies over all of the *refinements* of a Kripke model, where refinements can be said to correspond to the results of all possible informative updates.

In this paper we relate the action model quantifier to the refinement quantifier of van Ditmarsch and French [10]. In addition to our action model synthesis result, we show that the action model quantifier is equivalent to the refinement quantifier, and for the arbitrary action model logic we provide a sound and complete axiomatisation, show that it is expressively equivalent to modal logic, and show that it is decidable.

II. Technical Preliminaries

We recall a number of definitions from modal logic, the refinement modal logic of van Ditmarsch and French [10], and the action model logic of Baltag, Moss and Solecki [4].

Let P be a non-empty, countable set of propositional atoms, and let A be a non-empty, finite set of agents.

Definition II.1 (Kripke model)**.** A *Kripke model* $M = (S, R, V)$ consists of a *domain* S, which is a non-empty set of states (or possible worlds), an *accessibility* function $R : A \to \mathcal{P}(S \times S)$, which is a function from agents to accessibility relations on S, and a *valuation* function $V : S \to \mathcal{P}(P)$, which is a function from states to sets of propositional atoms.

The class of all Kripke models is called $\mathcal{K}$. A *pointed* Kripke model $M_s = (M, s)$ consists of a Kripke model $M = (S, R, V)$ along with a designated state $s \in S$.

We write R_a to denote $R(a)$. Given two states $s, t \in S$, we write sR_at to denote that $(s, t) \in R_a$. We write sR_a to denote the set of states $\{t \in S \mid sR_at\}$ and write R_at to denote the set of states $\{s \in S \mid sR_at\}$. As we will often be required to discuss several models at once, we will use the convention that $M = (S, R, V)$, $M' = (S', R', V')$, $M^\gamma = (S^\gamma, R^\gamma, V^\gamma)$, etc.

Definition II.2 (Bisimulation)**.** Let $M = (S, R, V)$ and $M' = (S', R', V')$ be Kripke models. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a *bisimulation* if and only if for every $a \in A$ and $(s, s') \in \mathfrak{R}$ the following, **atoms**, **forth-**a and **back-**a, holds:

atoms: $V(s) = V'(s')$

forth-a**:** for every $t \in sR_a$ there exists a $t' \in s'R'_a$ such that $(t, t') \in \mathfrak{R}$.

back-a**:** for every $t' \in s'R'_a$ there exists a $t \in sR_a$ such that $(t, t') \in \mathfrak{R}$.

If $(s, s') \in \mathfrak{R}$ then we call M_s and $M'_{s'}$ bisimilar, and write $M_s \leftrightarrows M'_{s'}$ to denote that there is a bisimulation between M_s and $M'_{s'}$.

Proposition II.1. *The relation $\leftrightarrows$ is an equivalence relation.*

This is a well-known result in modal logic; see Blackburn, de Rijke and Venema [6].

Definition II.3 (Simulation and refinement)**.** Let $B \subseteq A$ and let M and M' be Kripke models. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a B*-simulation* if and only if it satisfies **atoms**, **forth-**a for every $a \in A$ and **back-**a for every $a \in A \setminus B$.

If $(s, s') \in \mathfrak{R}$ then we call $M'_{s'}$ a B*-simulation* of M_s and call M_s a B*-refinement* of $M'_{s'}$. We write $M'_{s'} \rightrightarrows_B M_s$ or equivalently $M_s \leftleftarrows_B M'_{s'}$ to denote this.

In the case where $B = A$ we use the terms *simulation* and *refinement* in place of A-simulation and A-refinement, and we write $M'_{s'} \rightrightarrows M_s$ or equivalently $M_s \leftleftarrows M'_{s'}$. In the case where $B = \{a\}$ for some $a \in A$ we simply use the terms a*-simulation* and a*-refinement*, and we write $M'_{s'} \rightrightarrows_a M_s$ or equivalently $M_s \leftleftarrows_a M'_{s'}$.

Proposition II.2. *The relation $\leftleftarrows_B$ is a preorder.*

This is shown by van Ditmarsch and French [10].

Proposition II.3. *Let $B \subseteq A$ and let $M_s, M'_{s'}$ be Kripke models such that $M_s \leftrightarrows M'_{s'}$. Then $M'_{s'} \leftleftarrows_B M_s$.*

This follows trivially from the Definition II.2 and Definition II.3.

Definition II.4 (Action model)**.** Let $\mathcal{L}$ be a logical language. An action model $\mathsf{N} = (\mathsf{S}, \mathsf{R}, \mathsf{pre})$ with preconditions defined on $\mathcal{L}$ consists of a *domain* S, which is a non-empty, finite set of action points, an *accessibility* function $\mathsf{R} : A \to \mathcal{P}(\mathsf{S} \times \mathsf{S})$, which is a function from agents to accessibility relations on S, and a *precondition* function $\mathsf{pre} : S \to \mathcal{L}$, which is a function from action points to formulae from $\mathcal{L}$.

The class of all action models is called $\mathcal{AM}$. A *pointed* action model $\mathsf{N_u} = (\mathsf{N}, \mathsf{u})$ consists of an action model $\mathsf{N} = (\mathsf{S}, \mathsf{R}, \mathsf{pre})$ along with an action point $\mathsf{u} \in \mathsf{S}$. A *multi-pointed* action model $\mathsf{N_T} = (\mathsf{N}, \mathsf{T})$ consists of an action model $\mathsf{N} = (\mathsf{S}, \mathsf{R}, \mathsf{pre})$ along with a set of action points $\mathsf{T} \subseteq \mathsf{S}$.

We use the same abbreviations and conventions for action models as are used for Kripke models. We use the convention of using sans-serif fonts for action models, as in $\mathsf{N_u}$ and italic fonts for Kripke models, as in M_s.

III. Syntax and Semantics

Here we define the syntax and semantics of the arbitrary action model logic.

Definition III.1 (Language of arbitrary action model logic)**.** The language $\mathcal{L}_{\otimes\forall}$ of arbitrary action model logic is inductively defined as:

$$\phi \quad ::= \quad p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid [\mathsf{N_T}]\phi \mid \forall_B \phi$$

where $p \in P$, $a \in A$, $B \subseteq A$, and $\mathsf{N_T} \in \mathcal{AM}$ is a multi-pointed action model with preconditions defined on the language $\mathcal{L}_{\otimes\forall}$.

We use all of the standard abbreviations for propositional logic, in addition to the abbreviations $\Diamond_a \phi ::= \neg\Box_a\neg\phi$, $\langle \mathsf{N_T} \rangle \phi ::= \neg[\mathsf{N_T}]\neg\phi$ and $\exists_B \phi ::= \neg\forall_B\neg\phi$.

We refer to the language $\mathcal{L}_\otimes$ of action model logic, which is $\mathcal{L}_{\otimes\forall}$ without the $\forall_B$ operator, to the language $\mathcal{L}_\forall$ of refinement modal logic, which is $\mathcal{L}_{\otimes\forall}$ without the $[\alpha]$ operator, the language $\mathcal{L}$ of modal logic, which is $\mathcal{L}_\forall$ without the $\forall_B$ operator, and the language $\mathcal{L}_0$ of propositional logic, which is $\mathcal{L}$ without the $\Box_a$ operator. Note that for the sublanguage $\mathcal{L}_\otimes$ we restrict the language that the preconditions of action models are defined over to be $\mathcal{L}_\otimes$.

We also use the cover operator of Janin and Walukiewicz [17], following the definitions given by Bílková, Palmigiano, and Venema [5]. The cover operator, $\nabla_a \Gamma$ is an abbreviation defined by $\nabla_a \Gamma ::= \Box_a \bigvee_{\gamma\in\Gamma} \gamma \wedge \bigwedge_{\gamma\in\Gamma} \Diamond_a \gamma$, where Γ is a finite set of formulae. We note that the modal operators $\Box_a$, $\Diamond_a$ and ∇_a are interdefineable, as $\Box_a\phi \leftrightarrow \nabla_a\{\phi\} \vee \nabla_a\emptyset$ and $\Diamond_a\phi \leftrightarrow \nabla_a\{\phi, \top\}$. This is the basis of the axiomatisation for refinement modal logic, and plays an important part in our action model synthesis result. The cover operator allows us to define normal forms for modal logics that allow us to only consider conjunctions of modalities in specific situations when we define axiomatisations and provably correct translations. This was previously used as the basis of several axiomatisations of refinement modal logics [11], [15], [14], [16], [7].

The semantics of the arbitrary action model logic builds upon the semantics of the modal logic, the action model logic, and the refinement modal logic, so we define the semantics for these logics first.

Definition III.2 (Semantics of modal logic)**.** Let $M = (S, R, V) \in \mathcal{K}$. The interpretation of $\phi \in \mathcal{L}$ in the logic K is defined inductively as:

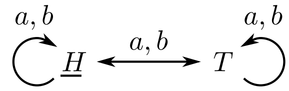

Fig. 1. The Kripke model M_H. Alice and Bob are initially uncertain about what the coin has landed on.

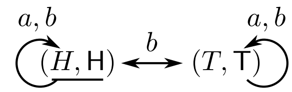

Fig. 2. The Kripke model $M_{(H,\mathsf{H})}$. After Alice has looked at the coin, she knows that the coin has landed heads up.

$$\begin{array}{rcl} M_s \models p & \text{iff} & p \in V(s) \\ M_s \models \neg\phi & \text{iff} & M_s \not\models \phi \\ M_s \models \phi \wedge \psi & \text{iff} & M_s \models \phi \text{ and } M_s \models \psi \\ M_s \models \Box_a \phi & \text{iff} & \text{for every } t \in sR_a\text{: } M_t \models \phi \end{array}$$

The Kripke semantics for modal logic are well-known; see Blackburn, de Rijke and Venema [6].

In epistemic logics, the formula $\Box_a\phi$ is often read as "a knows ϕ" or "a believes ϕ". Its dual $\Diamond_a\phi$ is often read as "a considers ϕ possible". We give an example of a multi-agent system involving knowledge.

We say that a formula ϕ is *satisfied* by a pointed Kripke model $M_s \in \mathcal{K}$ if and only if $M_s \models \phi$. We say that ϕ is satisfied by a Kripke model $M = (S, R, V) \in \mathcal{K}$ and we write $M \models \phi$ if and only if $M_s \models \phi$ for every $s \in S$. We say that ϕ is *valid* in a class of Kripke models $\mathcal{K}$ and we write $\mathcal{K} \models \phi$ if and only if $M \models \phi$ for every $M \in \mathcal{K}$.

Example III.1. Alice and Bob are about to play a nice game of chess, and have agreed to decide who moves first through a fair coin toss. Alice tosses the coin, but it lands too far away for either person to tell which side has landed face up.

If we let h stand for the proposition "The coin has landed heads up" (and the negation $\neg h$ stands for "The coin has landed tails up") then this situation is represented by the Kripke model $M = (S, R, V)$ where $S = \{H, T\}$, $R_a = R_b = \{(H, H), (H, T), (T, H), (T, T)\}$, $V(H) = \{h\}$ and $V(T) = \emptyset$. This is shown in Figure 1. The state M_H represents the possible world where the coin has landed heads up, whilst the state M_T represents the possible world where the coin has landed tails up.

Suppose that in reality the coin has landed heads up. Then the pointed Kripke model M_H represents the real world. About this situation we can say $M_H \models \Diamond_a h \wedge \Diamond_a \neg h$ meaning that Alice considers possible both that the coin may have landed heads up or that it may have landed tails up. Equivalently we can say $M_H \models \neg\Box_a h \wedge \neg\Box_a\neg h$ meaning that Alice does not know that the coin has landed heads up, nor does she know that the coin has landed tails up. We can also say $M_H \models \Box_b(\neg\Box_a h \wedge \neg\Box_a \neg h)$ meaning that Bob knows that Alice doesn't know whether the coin has landed heads up or tails up.

Definition III.3 (Semantics of action model logic)**.** Let $M = (S, R, V) \in \mathcal{K}$.

We first define *action model execution*. Let $\mathsf{N} \in \mathcal{AM}$ be an action model. We denote the result of executing the action model N on the Kripke model M as $M \otimes \mathsf{N}$, and we define the result as $M \otimes \mathsf{N} = M' = (S', R', V')$ where:

$$\begin{array}{rcl} S' & = & \{(s, \mathsf{u}) \mid s \in S, \mathsf{u} \in \mathsf{S}, M_s \vDash \mathsf{pre}(\mathsf{u})\} \\ (s, \mathsf{u}) R'_a (t, \mathsf{v}) & \text{iff} & s R_a t \text{ and } \mathsf{u} \mathsf{R}_a \mathsf{v} \\ V'((s, \mathsf{u})) & = & V(s) \end{array}$$

We also define *pointed action model execution* as $M_s \otimes \mathsf{N}_\mathsf{u} = (M \otimes \mathsf{N})_{(s,\mathsf{u})}$. Note that $M_s \otimes \mathsf{N}_\mathsf{u}$ is defined if and only if $(s, \mathsf{u}) \in S'$, if and only if $M_s \vDash \mathsf{pre}(\mathsf{u})$.

Then the interpretation of $\phi \in \mathcal{L}_\otimes$ in the logic $K_\otimes$ is the same as its interpretation in modal logic (Definition III.2), with the additional inductive cases:

$$\begin{array}{lll} M_s \vDash [\mathsf{N}_\mathsf{u}]\phi & \text{iff} & M_s \vDash \mathsf{pre}(\mathsf{u}) \text{ implies } M_s \otimes \mathsf{N}_\mathsf{u} \vDash \phi \\ M_s \vDash [\mathsf{N}_\mathsf{T}]\phi & \text{iff} & \text{for every } \mathsf{u} \in \mathsf{T} : M_s \vDash [\mathsf{N}_\mathsf{u}]\phi \end{array}$$

The semantics of action model logic are given by Baltag, Moss and Solecki [4].

The formula $[\mathsf{M}_\mathsf{T}]\phi$ can be read as "After successfully performing the informative update M_T, ϕ is true". Its dual $\langle \mathsf{M}_\mathsf{T} \rangle \phi$ can be read as "The informative update M_T can be successfully performed, and afterwards ϕ is true". We give another example involving changes of knowledge in a multi-agent system.

Example III.2. Continuing from Example III.1, suppose that Alice now decides to walk over to look at the coin, while Bob watches from a distance. We can represent the informative update of Alice looking at the coin by the action model $\mathsf{N} = (\mathsf{S}, \mathsf{R}, \mathsf{pre})$ where $\mathsf{S} = \{\mathsf{H}, \mathsf{T}\}$, $\mathsf{R}_a = \{(\mathsf{H}, \mathsf{H}), (\mathsf{T}, \mathsf{T})\}$, $\mathsf{R}_b = \{(\mathsf{H}, \mathsf{H}), (\mathsf{H}, \mathsf{T}), (\mathsf{T}, \mathsf{H}), (\mathsf{T}, \mathsf{T})\}$, $\mathsf{pre}(\mathsf{H}) = h$ and $\mathsf{pre}(\mathsf{T}) = \neg h$.

The result of performing the informative update N on the initial state M from Example III.1 is $M' = M \otimes \mathsf{N} = (S', R', V')$ where $S' = \{(H, \mathsf{H}), (T, \mathsf{T})\}$, $R'_a = \{((H, \mathsf{H}), (H, \mathsf{H})), ((T, \mathsf{T}), (T, \mathsf{T}))\}$, $R'_b = \{((H, \mathsf{H}), (H, \mathsf{H})), ((H, \mathsf{H}), (T, \mathsf{T})), ((T, \mathsf{T}), (H, \mathsf{H})), ((T, \mathsf{T}), (T, \mathsf{T}))\}$, $V'((H, \mathsf{H})) = \emptyset$ and $V'((T, \mathsf{T})) = \{h\}$. This is shown in Figure 2.

In reality the coin has landed heads up, so according to the preconditions of the action model the only action that could have occurred is that represented by the action point H. Then the pointed Kripke model $M'_{(H,\mathsf{H})}$ represents the real world after performing the informative update. We can make statements about what is true in this resulting state by using action model operators in the initial state. We can say $M_H \vDash [\mathsf{N}_\mathsf{H}]\Box_a h$ meaning that after the informative update, Alice knows that the coin has landed heads up. We can also say $M_H \vDash [\mathsf{N}_\mathsf{H}]\neg\Box_b h$, meaning that after the informative update Bob (still) does not know that the coin has landed heads up. We can also say that $M_H \vDash [\mathsf{N}_\mathsf{H}]\Box_b(\Box_a h \vee \Box_a \neg h)$ meaning that after the informative update Bob knows that Alice knows either that the coin has landed heads up or that it has landed tails up.

Baltag and Moss [3] provide many more examples of informative updates and their representations as action models.

Definition III.4 (Semantics of refinement modal logic)**.** Let $M = (S, R, V) \in \mathcal{K}$. The interpretation of $\phi \in \mathcal{L}_\forall$ in the logic $K_\forall$ is the same as its interpretation in modal logic (Definition III.2), with the additional inductive case:

$$\begin{array}{lll} M_s \vDash \forall_B \phi & \text{iff} & \text{for every } M'_{s'} \in \mathcal{K} \text{ such that} \\ & & M'_{s'} \preceq_B M_s : M'_{s'} \vDash \phi \end{array}$$

The semantics of refinement modal logic are given by van Ditmarsch and French [10]. We use an alternative but equivalent semantics in terms of B-refinements, first used by Hales, French and Davies [16].

The formula $\forall_B \phi$ can be read as "For every B-refinement, ϕ holds". Its dual $\exists_B \phi$ can be read as "There exists a B-refinement such that ϕ".

Example III.3. Continuing from Examples III.1 and III.2, we note that the Kripke model $M'_{(H,\mathsf{H})}$ is an a-refinement of M_H, i.e. $M'_{(H,\mathsf{H})} \preceq_a M_H$. We can make statements about the a-refinements of the initial state by using refinement quantifiers in the initial state. For example, we can say $M_H \vDash \exists_a \Box_a h$ meaning that there is an a-refinement of the initial state where $\Box_a h$ is true.

Before moving on to define the semantics of the arbitrary action model logic, we will make some remarks about the relationships between action models and refinement.

The refinement modal logic was introduced by van Ditmarsch and French [10] as a logic for quantifying over the results of informative updates. The statement $\exists \phi$ is intended to be read as "there exists an informative update after which ϕ is true". van Ditmarsch and French [10] give several results to justify the position that the refinements of a Kripke model correspond to the results of all possible informative updates. One such result is that the result of any action model execution is a refinement.

Proposition III.1. *Let $\mathsf{N}_\mathsf{u} \in \mathcal{AM}$ be an action model and let $M_s \in \mathcal{K}$ be a Kripke model such that $M_s \vDash \mathsf{pre}(\mathsf{u})$. Then $M_s \otimes \mathsf{N}_\mathsf{u} \preceq M_s$.*

This is shown by van Ditmarsch and French [10].

The converse also holds in the case of finite Kripke models: the refinements of a finite Kripke model correspond to the results of an action model execution.

Proposition III.2. *Let $M_s, M'_{s'} \in \mathcal{K}$ be Kripke models such that M_s is finite and $M'_{s'} \preceq M_s$. Then there exists an action model $\mathsf{N}_\mathsf{u} \in \mathcal{AM}$ such that $M_s \otimes \mathsf{N}_\mathsf{u} \leftrightarrows M'_{s'}$.*

This is also shown by van Ditmarsch and French [10]. We note however that we do not have this result in the case of refinements of *infinite* Kripke models.

Further to this, refinements represent monotonic changes in knowledge or belief, as do action models. In particular, positive formulae preserve truth under refinements of Kripke models.

Definition III.5 (Positive formulae)**.** Let $B \subseteq A$. The B-

positive formulae are defined inductively as:

$$\phi ::= p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box_a \phi \mid \Diamond_c \phi$$

where $p \in P$, $a \in A$ and $c \in A \setminus B$.

Proposition III.3. *Let $B \subseteq A$, let ϕ be a B-positive formula, and let $M_s, M'_{s'} \in \mathcal{K}$ such that $M'_{s'} \succeq_B M_s$. Then $M_s \models \phi$ implies $M'_{s'} \models \phi$.*

This is shown by van Ditmarsch and French [10] for the case of A-positive formulae. The generalisation to B-positive formulae where $B \subseteq A$ is straight-forward.

The intuition behind this result is that informative updates can only provide agents with new information and cannot revise information that agents already have. Therefore anything that an agent knows before an informative update, they should continue to know afterwards. This is not quite accurate in the case where an agent knows something about another agent's ignorance. For example, in the initial state of Example III.1, Bob knows that Alice doesn't know whether the coin has landed heads up or tails up. However after the informative update of Example III.2, Alice learns which side the coin landed on, and Bob learns that Alice knows which side the coin landed on, thus invalidating Bob's initial knowledge about Alice's ignorance. This result also gives a partial characterisation of the notion of a B-refinement as opposed to an (A-)refinement: a B-refinement corresponds to an informative update in which only agents in B may learn directly about the world, whilst agents not in B may only learn about the changes in knowledge of other agents. For example, in the initial state of Example III.1 Bob does not know that the coin has landed heads up, which we can denote by $M_H \models \neg\Box_b h$, or equivalently $M_H \models \Diamond_b \neg h$. As this is an a-positive formula, its truth is preserved by any a-refinement, and so it is not possible for Bob to learn about the state of the coin through any informative update that results in an a-refinement, such as the informative update from Example III.2.

We now introduce the semantics of arbitrary action model logic.

Definition III.6 (Semantics of arbitrary action model logic)**.** Let $M = (S, R, V) \in \mathcal{K}$. The interpretation of $\phi \in \mathcal{L}_{\otimes\forall}$ in the logic $K_{\otimes\forall}$ is the same as its interpretation in action model logic (Definition III.3), with the additional inductive case for the refinement quantifier of refinement modal logic (Definition III.4).

We note that, unintuitively, the semantics of Definition III.6 do not define the quantifier in terms of action models, but rather in terms of refinements. Semantics defined in terms of action models are as follows.

Definition III.7 (Restricted action models)**.** Let $B \subseteq A$ and let $\mathsf{N} \in \mathcal{AM}$. Then N is a *B-restricted action model* if and only if for every $\mathsf{u} \in \mathsf{S}$ and every $M_s \in \mathcal{K}$ such that $M_s \models \mathsf{pre}(\mathsf{u})$ we have $M_s \otimes \mathsf{N_u} \succeq_B M_s$. We call the class of B-restricted action models $\mathcal{AM}_B$.

Definition III.8 (Alternative semantics of arbitrary action model logic)**.** Let $M = (S, R, V) \in \mathcal{K}$. The interpretation of $\phi \in \mathcal{L}_{\otimes\forall}$ in the logic $K_{\otimes\forall}$ is the same as its interpretation in action model logic (Definition III.3), with the additional inductive case:

$$M_s \models \forall_B \phi \quad \text{iff} \quad \text{for every } \mathsf{N_u} \in \mathcal{AM}_B : M_s \models [\mathsf{N_u}]\phi$$

Our eventual goal is to show that the semantics of Definitions III.6 and III.8 are equivalent for the class $\mathcal{K}$ of all Kripke models. However for now we use the semantics of Definition III.6, that define the quantifier in terms of refinements. In Section IV this allows us to use existing results from refinement modal logic to provide a sound and complete axiomatisation, along with expressivity and decidability results. We use these results in Section V to provide our action model synthesis results. A corollary of the action model synthesis result will then be that the semantics of Definition III.6 and Definition III.8 are equivalent.

Finally we give some results that we use later.

Proposition III.4. *Let $\phi \in \mathcal{L}$ and let $M_s, M'_{s'} \in \mathcal{K}$ such that $M_s \leftrightarrows M'_{s'}$. Then $M_s \models \phi$ if and only if $M'_{s'} \models \phi$.*

This is a well-known result in modal logic; see Blackburn, de Rijke and Venema [6].

Definition III.9 (Bisimulation of action models)**.** Let $\mathsf{N} = (\mathsf{S}, \mathsf{R}, \mathsf{pre})$ and $\mathsf{N}' = (\mathsf{S}', \mathsf{R}', \mathsf{pre}')$ be action models with preconditions defined on $\mathcal{L}_\otimes$. A non-empty relation $\mathfrak{R} \subseteq \mathsf{S} \times \mathsf{S}'$ is a *bisimulation* if and only if for every $a \in A$ and $(\mathsf{u}, \mathsf{u}') \in \mathfrak{R}$ the following, **pre**, **forth-**a and **back-**a, holds:

pre: $\vdash \mathsf{pre}(\mathsf{u}) \leftrightarrow \mathsf{pre}'(\mathsf{u}')$

forth-a**:** for every $\mathsf{v} \in \mathsf{uR}_a$ there exists a $\mathsf{v}' \in \mathsf{u}'\mathsf{R}'_a$ such that $(v, v') \in \mathfrak{R}$.

back-a**:** for every $\mathsf{t}' \in \mathsf{u}'\mathsf{R}'_a$ there exists a $\mathsf{v} \in \mathsf{uR}_a$ such that $(\mathsf{v}, \mathsf{v}') \in \mathfrak{R}$.

If $(\mathsf{u}, \mathsf{u}') \in \mathfrak{R}$ then we call $\mathsf{N_u}$ and $\mathsf{N}'_{\mathsf{u}'}$ bisimilar, and write $\mathsf{N_u} \leftrightarrows \mathsf{N}'_{\mathsf{u}'}$ to denote that there is a bisimulation between $\mathsf{N_u}$ and $\mathsf{N}'_{\mathsf{u}'}$.

This definition is given by van Ditmarsch, van der Hoek and Kooi [12, p. 158].

Proposition III.5. *Let $\mathsf{N_u}, \mathsf{N}'_{\mathsf{u}'} \in \mathcal{AM}$ be action models such that $\mathsf{N_u} \leftrightarrows \mathsf{N}'_{\mathsf{u}'}$, and let $M_s \in \mathcal{K}$ be a Kripke model such that $M_s \models \mathsf{pre}(\mathsf{u})$. Then $M_s \otimes \mathsf{N_u} \leftrightarrows M_s \otimes \mathsf{N}'_{\mathsf{u}'}$.*

This is shown by van Ditmarsch, van der Hoek and Kooi [12, p. 158].

Definition III.10 (Cover disjunctive normal form)**.** A formula in *cover disjunctive normal form* is defined inductively by:

$$\alpha ::= \pi \wedge \bigwedge_{b \in B} \nabla_b \Gamma_b \mid \alpha \vee \alpha$$

where $\pi \in \mathcal{L}_0$, $B \subseteq A$ and for every $b \in B$, Γ_b is a finite set of formulae in cover disjunctive normal form.

Proposition III.6. *Every formula of $\mathcal{L}$ is equivalent to a formula in cover disjunctive normal form.*

This is shown by Hales [14] and Bozzelli, et al. [7]. The cover logic disjunctive normal form is used in the completeness proofs of Section IV and in the action model synthesis result of Section V.

IV. Axiomatisation

In this section we provide a sound and complete axiomatisation for the arbitrary action model logic $K_{\otimes\forall}$. We first recall the well-known axiomatisation $\mathbf{K}$ for modal logic *K*, the axiomatisation $\mathbf{AML_K}$ for action model logic $K_\otimes$, given by Baltag, Moss and Solecki [4], and the axiomatisation $\mathbf{RML_K}$ for refinement modal logic $K_\forall$, given by Hales [14] and Bozzelli, et al. [7].

Definition IV.1 (Axiomatisation $\mathbf{K}$)**.** The axiomatisation $\mathbf{K}$ is a substitution schema consisting of the following axioms:

$$\begin{array}{rl} \mathbf{P} & \text{All propositional tautologies} \\ \mathbf{K} & \Box_a(\phi \to \psi) \to (\Box_a\phi \to \Box_a\psi) \end{array}$$

Along with the rules:

$$\begin{array}{rl} \mathbf{MP} & \text{From } \vdash \phi \to \psi \text{ and } \vdash \phi\text{, infer } \vdash \psi \\ \mathbf{NecK} & \text{From } \vdash \phi \text{ infer } \vdash \Box_a\phi \end{array}$$

We say that a formula ϕ is *provable* under an axiomatisation, and we write $\vdash \phi$ if and only if it can be derived using some finite sequence of axioms and rules from that axiomatisation. When we are discussing provability, it should be clear from context which axiomatisation we are using.

Lemma IV.1. *The axiomatisation* $\mathbf{K}$ *is sound and complete with respect to the logic K.*

Soundness and completeness of $\mathbf{K}$ are well-known results; see Blackburn, de Rijke and Venema [6].

Definition IV.2 (Axiomatisation $\mathbf{AML_K}$)**.** The axiomatisation $\mathbf{AML_K}$ is a substitution schema consisting of the rules and axioms of $\mathbf{K}$ along with the axioms:

$$\begin{array}{rl} \mathbf{AP} & [\mathsf{N_u}]\pi \leftrightarrow (\mathsf{pre}(\mathsf{u}) \to \pi) \text{ for } \pi \in \mathcal{L}_0 \\ \mathbf{AN} & [\mathsf{N_u}]\neg\phi \leftrightarrow (\mathsf{pre}(\mathsf{u}) \to \neg[\mathsf{N_u}]\phi) \\ \mathbf{AC} & [\mathsf{N_u}](\phi \wedge \psi) \leftrightarrow ([\mathsf{N_u}]\phi \wedge [\mathsf{N_u}]\psi) \\ \mathbf{AK} & [\mathsf{N_u}]\Box_a\phi \leftrightarrow (\mathsf{pre}(\mathsf{u}) \to \bigwedge_{\mathsf{v}\in\mathsf{uR}_a} \Box_a[\mathsf{N_v}]\phi) \\ \mathbf{AU} & [\mathsf{N_T}]\phi \leftrightarrow \bigwedge_{\mathsf{u}\in\mathsf{T}}[\mathsf{N_u}]\phi \end{array}$$

and the rule:

$$\mathbf{NecA} \quad \text{From } \vdash \phi \text{ infer } \vdash [\mathsf{N_u}]\phi$$

Proposition IV.2. *The axiomatisation* $\mathbf{AML_K}$ *is sound and complete with respect to the logic* $K_\otimes$.

Soundness and completeness are shown by Baltag, Moss and Solecki [4]. The completeness of $\mathbf{AML_K}$ is shown via a provably correct translation from $\mathcal{L}_\otimes$ to the sublanguage of $\mathcal{L}$. As the axiomatisation $\mathbf{AML_K}$ contains the rules and axioms of $\mathbf{K}$, which are complete with respect to the class of Kripke models for formulae in $\mathcal{L}$, this shows the completeness of $\mathbf{AML_K}$.

The translation is performed on a formula by iteratively selecting subformulae of the form $[\mathsf{N_T}]\phi$, where $\phi \in \mathcal{L}$. The axiom $\mathbf{AU}$ is used to expand multi-pointed action model executions into single-pointed action model executions, the axioms $\mathbf{AN}$, $\mathbf{AC}$ and $\mathbf{AK}$ are used to push the $[\mathsf{N_s}]$ operators inside negations, conjunctions and modalities, and the axiom $\mathbf{AP}$ is used to remove $[\mathsf{N_s}]$ operators once they are only applied to propositional formulae. This process is repeated until there are no $[\mathsf{N_T}]$ operators left in the formula.

Definition IV.3 (Axiomatisation $\mathbf{RML_K}$)**.** The axiomatisation $\mathbf{RML_K}$ is a substitution schema consisting of the rules and axioms of $\mathbf{K}$ along with the axioms:

$$\begin{array}{rl} \mathbf{R} & \forall_B(\phi \to \psi) \to (\forall_B\phi \to \forall_B\psi) \\ \mathbf{RP} & \forall_B\pi \leftrightarrow \pi \text{ where } \pi \text{ is a propositional formula} \\ \mathbf{RK} & \exists_B\nabla_a\Gamma_a \leftrightarrow \bigwedge_{\gamma\in\Gamma} \Diamond_a\exists_B\gamma \text{ where } a \in B \\ \mathbf{RComm} & \exists_B\nabla_a\Gamma_a \leftrightarrow \nabla_a\{\exists_B\gamma \mid \gamma \in \Gamma_a\} \text{ where } a \notin B \\ \mathbf{RDist} & \exists_B \bigwedge_{c\in C} \nabla_c\Gamma_c \leftrightarrow \bigwedge_{c\in C} \exists_B\nabla_c\Gamma_c \text{ where } C \subseteq A \end{array}$$

and the rule:

$$\mathbf{NecR} \quad \text{From } \vdash \phi \text{ infer } \vdash \forall_B\phi$$

Proposition IV.3. *The axiomatisation* $\mathbf{RML_K}$ *is sound and complete with respect to the logic* $K_\forall$.

Soundness and completeness are shown by Hales [14] and Bozzelli, et al. [7]. Similar to the axiomatisation $\mathbf{AML_K}$, the completeness of $\mathbf{RML_K}$ is shown via a provably correct translation from $\mathcal{L}_\forall$ to the sublanguage of $\mathcal{L}$.

As the sublanguage of $\mathcal{L}$ is complete with respect to the class of Kripke models using the axioms and rules of $\mathbf{K}$, which are also included in $\mathbf{RML_K}$, this shows the completeness of $\mathbf{RML_K}$. As the axiomatisation $\mathbf{RML_K}$ contains the rules and axioms of $\mathbf{K}$, which are complete with respect to the class of Kripke models for formulae in $\mathcal{L}$, this shows the completeness of $K_\forall$.

The translation is performed on a formula by iteratively selecting subformulae of the form $\exists_B\phi$, where $\phi \in \mathcal{L}$. The subformula ϕ is converted to cover disjunctive normal form, and then the axioms $\mathbf{R}$, $\mathbf{RP}$, $\mathbf{RK}$, $\mathbf{RComm}$ and $\mathbf{RDist}$ are used to push the $\exists_B$ operator inwards, inside disjunctions, conjunctions and cover operators. Once all $\exists_B$ operators are applied only to propositional formulae, $\mathbf{RP}$ is used to remove these operators from the formula. This process is repeated until there are no $\exists_B$ operators left in the formula.

We now provide an axiomatisation for $K_{\otimes\forall}$.

Definition IV.4 (Axiomatisation $\mathbf{AAML_K}$)**.** The axiomatisation $\mathbf{AAML_K}$ is a substitution schema consisting of the rules and axioms of the axiomatisations $\mathbf{RML_K}$ and $\mathbf{AML_K}$.

Theorem IV.4. *The axiomatisation* $\mathbf{AAML_K}$ *is sound and complete with respect to the logic* $K_{\otimes\forall}$.

Proof: The soundness of the rules of $\mathbf{AAML_K}$ are trivial to show. The soundness of the axioms of $\mathbf{AAML_K}$, in the restricted cases where substitutions in the axioms only occur with formulae from $\mathcal{L}$ and action models may only have preconditions from $\mathcal{L}$, follows from the soundness of the axioms of $\mathbf{RML_K}$ and $\mathbf{AML_K}$.

We note that these restricted axioms give us enough for a provably correct translation from the full language of $\mathcal{L}_{\otimes\forall}$ to $\mathcal{L}$, by utilising the provably correct translations used in the completeness proofs for $\mathbf{AML_K}$ and $\mathbf{RML_K}$. We proceed by iteratively selecting subformulae (including subformulae of preconditions of the action models listed in the formula) of the form $[\mathsf{N_T}]\phi$ or $\exists_B\phi$, where $\mathsf{N_T}$ only has preconditions from $\mathcal{L}$, and where $\phi \in \mathcal{L}$. We then use the appropriate provably correct translation for $K_\otimes$ or $K_\forall$ to translate this subformula into a formula from $\mathcal{L}$. This process is repeated until there are no $\exists_B$ or $[\mathsf{N_T}]$ operators left in the formula. This provably correct translation from $\mathcal{L}_{\otimes\forall}$ to $\mathcal{L}$ allows us to show the soundness of the unrestricted versions of the $\mathbf{RML_K}$ and $\mathbf{AML_K}$ axioms, as well as to show the completness of these axioms. ■

Using the provably correct translation above we get the following corollaries.

Corollary IV.5. *The logics $K_{\otimes\forall}$ and K are expressively equivalent.*

Corollary IV.6. *The logic $K_{\otimes\forall}$ is decidable.*

We note that as in the provably correct translation for refinement modal logic [11], the translation for arbitrary action model logic may result in a non-elementary increase in the size of the formula. A decision procedure relying on this translation would therefore have a non-elementary complexity.

V. Synthesis of Action Models

In this section we present the main technical result of the paper: we aim, for a given post-condition ϕ, to find an action model $\mathsf{N_T}$ that will result in ϕ becoming true when it is executed on any Kripke model. However in general, whether such an action model exists and will be successful depends on the post-condition and on the particular Kripke model that it will be executed on. For example, it should be clear that no action model can result in $\bot$ as a post-condition, no matter what Kripke model it is executed on. As another example, from Propositions III.1 and III.3 it should be clear that no action model can result in $\neg\Box_a p$ if it is executed on a Kripke model that has $\Box_a p$ initially. Therefore we aim to synthesise an action model that will result in the given post-condition whenever it is possible to do so.

More precisely: let $B \subseteq A$ and let $\phi \in \mathcal{L}_{\otimes\forall}$. Then we aim to find an action model $\mathsf{N_T} \in \mathcal{AM}_B$ such that for every $M_s \in \mathcal{K}$: if there exists $\mathsf{N'_{T'}} \in \mathcal{AM}_B$ such that $M_s \models \langle \mathsf{N'_{T'}}\rangle\phi$, then $M_s \models \langle\mathsf{N_T}\rangle\phi$. In other words, for any initial Kripke model, if any B-restricted action model can result in ϕ then the action model we have synthesised will result in ϕ.

The result that we show is actually a stronger result in terms of refinement quantifiers. We aim to find an action model $\mathsf{N_T} \in \mathcal{AM}_B$ such that for every $M_s \in \mathcal{K}$: if $M_s \models \exists_B\phi$ then $M_s \models \langle\mathsf{N_T}\rangle\phi$. In other words, for any initial Kripke model, if there is a B-refinement such that ϕ, then the B-restricted action model we have synthesised will result in ϕ. This result is stronger because we have not already showed that the existence of a B-refinement such that ϕ implies the existence of a B-restricted action model that results in ϕ; the correspondence from refinements to action model executions of Proposition III.2 only applies for refinements of finite Kripke models, but we have made no such restriction here. A corollary of this result is that the semantics of the arbitrary action model logic defined in terms of the refinement quantifier (Definition III.6) is equivalent to the semantics in terms of the action model quantifier (Definition III.8).

We use the expressivity result for $K_{\otimes\forall}$ and the cover disjunctive normal form to assist with our action model synthesis result. From Theorem IV.4, for any post-condition from $\mathcal{L}_{\otimes\forall}$ we have an equivalent in $\mathcal{L}$. From Proposition III.6, for any formula in $\mathcal{L}$ we have an equivalent in cover disjunctive normal form. We therefore proceed by induction on the structure of the cover disjunctive normal formula to show that we can construct an action model that results in the desired post-condition, whenever it is possible. To these ends we introduce two lemmas, each dealing with a different inductive case.

Lemma V.1. *Let $B \subseteq A$, let $\phi = \alpha \vee \beta \in \mathcal{L}_{\otimes\forall}$, and let $\mathsf{N}^\alpha_{\mathsf{T}^\alpha} = ((\mathsf{S}^\alpha, \mathsf{R}^\alpha, \mathsf{pre}^\alpha), \mathsf{T}^\alpha), \mathsf{N}^\beta_{\mathsf{T}^\beta} = ((\mathsf{S}^\beta, \mathsf{R}^\beta, \mathsf{pre}^\beta), \mathsf{T}^\alpha) \in \mathcal{AM}$ be multi-pointed B-restricted action models such that for $\gamma \in \{\alpha,\beta\}$: $\vdash [\mathsf{N}^\gamma_{\mathsf{T}^\gamma}]\gamma$ and $\vdash \langle\mathsf{N}^\gamma_{\mathsf{T}^\gamma}\rangle\gamma \leftrightarrow \exists_B\gamma$. Then there exists a multi-pointed B-restricted action model $\mathsf{N_T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ such that $\vdash [\mathsf{N_T}]\phi$ and $\vdash \langle\mathsf{N_T}\rangle\phi \leftrightarrow \exists_B\phi$.*

Proof: Without loss of generality we assume that S^α and S^β are disjoint.

We construct the multi-pointed action model $\mathsf{N_T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ where:

$$\begin{aligned}\mathsf{S} &= \mathsf{S}^\alpha \cup \mathsf{S}^\beta\\ \mathsf{R} &= \mathsf{R}^\alpha \cup \mathsf{R}^\beta\\ \mathsf{pre} &= \mathsf{pre}^\alpha \cup \mathsf{pre}^\beta\\ \mathsf{T} &= \mathsf{T}^\alpha \cup \mathsf{T}^\beta\end{aligned}$$

We note that as N is formed by the disjoint union of N^α and N^β, then each action point of N^α and N^β is bisimilar to the corresponding action point from N.

First we show that $\vdash [\mathsf{N_T}]\phi$.

$$\begin{aligned}&\vdash \quad [\mathsf{N}^\alpha_{\mathsf{T}^\alpha}]\alpha \wedge [\mathsf{N}^\beta_{\mathsf{T}^\beta}]\beta && (1)\\ &\vdash \quad [\mathsf{N}_{\mathsf{T}^\alpha}]\alpha \wedge [\mathsf{N}_{\mathsf{T}^\beta}]\beta && (2)\\ &\vdash \quad [\mathsf{N}_{\mathsf{T}^\alpha}](\alpha \vee \beta) \wedge [\mathsf{N}_{\mathsf{T}^\beta}](\alpha \vee \beta) && (3)\\ &\vdash \quad [\mathsf{N_T}](\alpha \vee \beta) && (4)\end{aligned}$$

(1) follows from hypothesis, (2) follows from Proposition III.5 and the fact that $\mathsf{N}^\alpha_{\mathsf{T}^\alpha}$ is bisimilar to $\mathsf{N}_{\mathsf{T}^\alpha}$, and $\mathsf{N}^\beta_{\mathsf{T}^\beta}$ is bisimilar to $\mathsf{N}_{\mathsf{T}^\beta}$, (3) is simple disjunction introduction, and (4) follows from **AU**, as $\mathsf{T} = \mathsf{T}^\alpha \cup \mathsf{T}^\beta$.

Next we show that N is a B-restricted action model. Let $\gamma \in \{\alpha, \beta\}$, let $\mathsf{u} \in \mathsf{S}^\gamma$ and let $M_s \in \mathcal{K}$. As N^γ is a B-restricted action model, then $M_s \otimes \mathsf{N}^\gamma_\mathsf{u} \succeq_B M_s$. As N is formed from the disjoint union of N^α and N^β then it is a simple matter to show that $\mathsf{N_u} \leftrightarrows \mathsf{N}^\gamma_\mathsf{u}$. Therefore from Proposition III.5 we have that $M_s \otimes \mathsf{N_u} \leftrightarrows M_s \otimes \mathsf{N^\gamma_u}$, which from Proposition II.3 means that $M_s \otimes \mathsf{N_u} \leftleftarrows_B M_s \otimes \mathsf{N^\gamma_u}$. From Proposition II.2, the $\leftleftarrows_B$ relation is transitive, therefore $M_s \otimes \mathsf{N_u} \leftleftarrows_B M_s$, and so N is a B-restricted action model.

Finally we show that $\vdash \langle \mathsf{N_T} \rangle \phi \leftrightarrow \exists_B \phi$.

$$\begin{array}{lll}\vdash & \exists_B(\alpha \vee \beta) \to (\exists_B \alpha \vee \exists_B \beta) & (5)\\ \vdash & \exists_B(\alpha \vee \beta) \to (\langle \mathsf{N}^\alpha_{\mathsf{T}^\alpha} \rangle \alpha \vee \langle \mathsf{N}^\beta_{\mathsf{T}^\beta} \rangle \beta) & (6)\\ \vdash & \exists_B(\alpha \vee \beta) \to (\langle \mathsf{N}_{\mathsf{T}^\alpha} \rangle \alpha \vee \langle \mathsf{N}_{\mathsf{T}^\beta} \rangle \beta) & (7)\\ \vdash & \exists_B(\alpha \vee \beta) \to (\langle \mathsf{N}_{\mathsf{T}^\alpha} \rangle (\alpha \vee \beta) \vee \langle \mathsf{N}_{\mathsf{T}^\beta} \rangle (\alpha \vee \beta)) & (8)\\ \vdash & \exists_B(\alpha \vee \beta) \to (\langle \mathsf{N_T} \rangle (\alpha \vee \beta)) & (9)\end{array}$$

(5) follows from the $\mathbf{AAML_K}$ axiom **R**, (6) follows from hypothesis, (7) follows from Proposition III.5 and the fact that $\mathsf{N}^\alpha_{\mathsf{T}^\alpha}$ is bisimilar to $\mathsf{N}_{\mathsf{T}^\alpha}$, and $\mathsf{N}^\beta_{\mathsf{T}^\beta}$ is bisimilar to $\mathsf{N}_{\mathsf{T}^\beta}$, (8) is simple disjunction introduction, and (9) follows from the $\mathbf{AAML_K}$ axiom **AU**. The converse, $\vdash \langle \mathsf{N_T} \rangle \phi \to \exists_B \phi$ follows from a simple semantic argument, from Proposition III.1, the fact that $\mathsf{N_T}$ is a B-restricted action model, and from the completeness of $\mathbf{AAML_K}$. ■

Lemma V.2. *Let $B, C \subseteq A$, let $\phi = \pi \wedge \bigwedge_{c \in C} \nabla_c \Gamma_c \in \mathcal{L}_{\otimes\forall}$ and for every $c \in C$ and $\gamma \in \Gamma_c$ let $\mathsf{N}^\gamma_{\mathsf{T}^\gamma} = ((\mathsf{S}^\gamma, \mathsf{R}^\gamma, \mathsf{pre}^\gamma), \mathsf{T}^\gamma) \in \mathcal{AM}$ be a multi-pointed B-restricted action model such that $\vdash [\mathsf{N}^\gamma_{\mathsf{T}^\gamma}]\gamma$ and $\vdash \langle \mathsf{N}^\gamma_{\mathsf{T}^\gamma} \rangle \gamma \leftrightarrow \exists_B \gamma$. Then there exists a multi-pointed B-restricted action model $\mathsf{N_T} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{T})$ such that $\vdash [\mathsf{N_T}]\phi$ and $\vdash \langle \mathsf{N_T} \rangle \phi \leftrightarrow \exists_B \phi$.*

Proof: Without loss of generality we assume that each of the S^γ are disjoint.

We construct the action model $\mathsf{N_u} = ((\mathsf{S}, \mathsf{R}, \mathsf{pre}), \mathsf{u})$ where:

$$\begin{array}{lll}\mathsf{S} & = & \{\mathsf{u}, \mathsf{skip}\} \cup \bigcup_{c \in C, \gamma \in \Gamma_c} \mathsf{S}^\gamma\\ \mathsf{R}_a & = & \{(\mathsf{u}, \mathsf{v}) \mid \gamma \in \Gamma_a, \mathsf{v} \in \mathsf{T}^\gamma\} \cup \{(\mathsf{skip}, \mathsf{skip})\} \cup \\ & & \bigcup_{c \in C, \gamma \in \Gamma_c} \mathsf{R}^\gamma_a \text{ for } a \in C\\ \mathsf{R}_a & = & \{(\mathsf{u}, \mathsf{skip}), (\mathsf{skip}, \mathsf{skip})\} \cup \bigcup_{c \in C, \gamma \in \Gamma_c} \mathsf{R}^\gamma_a \text{ for } a \notin C\\ \mathsf{pre} & = & \{(\mathsf{u}, \exists_B \phi), (\mathsf{skip}, \top)\} \cup \bigcup_{c \in C, \gamma \in \Gamma_b} \mathsf{pre}^\gamma\end{array}$$

We note that as N is formed by taking the disjoint union of each action model N^γ, and the only edges added to states in each N^γ are inward edges, then each action point of N^γ is bisimilar to the corresponding action point from N.

We will show that $\vdash [\mathsf{N_u}]\phi$ in several parts. We note that from the definition of the cover operator:

$$\phi \equiv \pi \wedge \bigwedge_{c \in C} (\Box_c \bigvee_{\gamma \in \Gamma_c} \gamma \wedge \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \gamma)$$

Therefore we will show individually that:

1) $\vdash [\mathsf{N_u}]\pi$
2) $\vdash [\mathsf{N_u}]\Box_c \bigvee_{\gamma \in \Gamma_c} \gamma$ for every $c \in C$; and
3) $\vdash [\mathsf{N_u}] \bigwedge_{\gamma \in \Gamma} \Diamond_c \gamma$ for every $c \in C$

and then these results in the end.

First we show that $\vdash [\mathsf{N_u}]\pi$.

$$\begin{array}{lll}\vdash & \phi \to \pi & (10)\\ \vdash & \neg\pi \to \neg\phi & (11)\\ \vdash & \forall_B(\neg\pi \to \neg\phi) & (12)\\ \vdash & \forall_B\neg\pi \to \forall_B\neg\phi & (13)\\ \vdash & \neg\forall_B\neg\phi \to \neg\forall_B\neg\pi & (14)\\ \vdash & \exists_B\phi \to \exists_B\pi & (15)\\ \vdash & \mathsf{pre}(\mathsf{u}) \to \pi & (16)\\ \vdash & [\mathsf{N_u}]\pi & (17)\end{array}$$

(12) follows from **NecR**, (13) follows from **R**, (15) follows from the definition of $\exists_B$, (16) follows from the definition of N, as $\mathsf{pre}(\mathsf{u}) = \exists_B\phi$, (17) follows from **AP**, and the rest is straight-forward. Therefore we have that $\vdash [\mathsf{N_u}]\pi$.

Next we show that $\vdash [\mathsf{N_u}]\Box_c \bigvee_{\gamma \in \Gamma_c} \gamma$ for every $c \in C$. Let $c \in C$. Then:

$$\begin{array}{lll}\vdash & [\mathsf{N}^\gamma_{\mathsf{T}^\gamma}]\gamma \text{ for every } \gamma \in \Gamma_c & (18)\\ \vdash & \bigwedge_{\mathsf{v} \in \mathsf{T}^\gamma} [\mathsf{N}^\gamma_\mathsf{v}]\gamma \text{ for every } \gamma \in \Gamma_c & (19)\\ \vdash & \Box_c \bigwedge_{\mathsf{v} \in \mathsf{T}^\gamma} [\mathsf{N}^\gamma_\mathsf{v}]\gamma \text{ for every } \gamma \in \Gamma_c & (20)\\ \vdash & \bigwedge_{\mathsf{v} \in \mathsf{T}^\gamma} \Box_c[\mathsf{N}^\gamma_\mathsf{v}]\gamma \text{ for every } \gamma \in \Gamma_c & (21)\\ \vdash & \bigwedge_{\gamma \in \Gamma_c} \bigwedge_{\mathsf{v} \in \mathsf{T}^\gamma} \Box_c[\mathsf{N}^\gamma_\mathsf{v}]\gamma & (22)\\ \vdash & \bigwedge_{\gamma \in \Gamma_c} \bigwedge_{\mathsf{v} \in \mathsf{T}^\gamma} \Box_c[\mathsf{N}^\gamma_\mathsf{v}] \bigvee_{\gamma' \in \Gamma_c} \gamma' & (23)\\ \vdash & \bigwedge_{\mathsf{v} \in \mathsf{uR}_c} \Box_c[\mathsf{N_v}] \bigvee_{\gamma' \in \Gamma_c} \gamma' & (24)\\ \vdash & \mathsf{pre}(\mathsf{u}) \to \bigwedge_{\mathsf{v} \in \mathsf{uR}_c} \Box_c[\mathsf{N_v}] \bigvee_{\gamma' \in \Gamma_c} \gamma' & (25)\\ \vdash & [\mathsf{N_u}]\Box_c \bigvee_{\gamma' \in \Gamma_c} \gamma' & (26)\end{array}$$

(18) follows from hypothesis, (19) follows from **AU**, (24) follows from the definition of N, (26) follows from **AK**, and the rest is straight-forward. Therefore we have that $\vdash [\mathsf{N_u}]\Box_c \bigvee_{\gamma' \in \Gamma_c} \gamma'$ for every $c \in C$.

Next we show that $\vdash [\mathsf{N_u}] \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \gamma$ for every $c \in C$. Let $c \in C$.

Suppose that $c \in B$. Then:

$$\begin{array}{lll}\vdash & \exists_B\phi \to \exists_B \nabla_c \Gamma_c & (27)\\ \vdash & \exists_B\phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \exists_B \gamma & (28)\end{array}$$

(27) follows from similar reason to (10-15) above and (28) follows from **RK**.

Suppose that $c \notin B$. Then:

$$\vdash \quad \exists_B \phi \to \exists_B \nabla_c \Gamma_c \tag{29}$$
$$\vdash \quad \exists_B \phi \to \nabla_c \{\exists_B \gamma \mid \gamma \in \Gamma_c\} \tag{30}$$
$$\vdash \quad \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \exists_B \gamma \tag{31}$$

(29) follows from similar reasoning to (10-15) above, (30) follows from **RComm**, and (31) follows from the definition of the cover operator.

Therefore we have for every $c \in C$ that $\vdash \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \exists_B \gamma$. Then for every $c \in C$:

$$\vdash \quad \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \exists_B \gamma \tag{32}$$
$$\vdash \quad \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \langle \mathsf{N}^\gamma_{\mathsf{T}^\gamma} \rangle \gamma \tag{33}$$
$$\vdash \quad \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \bigvee_{\mathsf{v} \in \mathsf{T}^\gamma} \langle \mathsf{N}^\gamma_\mathsf{v} \rangle \gamma \tag{34}$$
$$\vdash \quad \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \bigvee_{\mathsf{v} \in \mathsf{T}^\gamma} \langle \mathsf{N}_\mathsf{v} \rangle \gamma \tag{35}$$
$$\vdash \quad \exists_B \phi \to \bigwedge_{\gamma \in \Gamma_c} \Diamond_c \bigvee_{\mathsf{v} \in \mathsf{uR}_c} \langle \mathsf{N}_\mathsf{u} \rangle \gamma \tag{36}$$
$$\vdash \quad \bigwedge_{\gamma \in \Gamma_c} \left(\exists_B \phi \to \bigvee_{\mathsf{v} \in \mathsf{vR}_c} \Diamond_c \langle \mathsf{N}_\mathsf{v} \rangle \gamma \right) \tag{37}$$
$$\vdash \quad \bigwedge_{\gamma \in \Gamma_c} (\exists_B \phi \to \langle \mathsf{N}_\mathsf{u} \rangle \Diamond_c \gamma) \tag{38}$$
$$\vdash \quad \bigwedge_{\gamma \in \Gamma_c} [\mathsf{N}_\mathsf{u}] \Diamond_c \gamma \tag{39}$$

(32) follows from (28) and (31) from above, (33) follows from hypothesis, (34) follows from **AU**, (35) and (36) follow from the definitions of N, and (38) follows from **AK**. Therefore we have that $\vdash \bigwedge_{\gamma \in \Gamma_c} [\mathsf{N}_\mathsf{u}] \Diamond_c \gamma$ for every $c \in C$.

Finally we combine the results we have shown so far:

$$\vdash \quad [\mathsf{N}_\mathsf{u}]\pi \wedge \bigwedge_{c \in C} \left([\mathsf{N}_\mathsf{u}] \Box_c \bigvee_{\gamma \in \Gamma_c} \gamma \wedge \bigwedge_{\gamma \in \Gamma_c} [\mathsf{N}_\mathsf{u}] \Diamond_c \gamma \right) \tag{40}$$
$$\vdash \quad [\mathsf{N}_\mathsf{u}](\pi \wedge \bigwedge_{c \in C} \nabla_c \Gamma_c) \tag{41}$$

(40) follows from (14), (26) and (38) above, and (41) follows from **AC**. Therefore we have that $\vdash [\mathsf{N}_\mathsf{u}](\pi \wedge \bigwedge_{c \in C} \nabla_c \Gamma_c)$.

Next we show that $\vdash \langle \mathsf{N}_\mathsf{u} \rangle \phi \leftrightarrow \exists_B \phi$. This is straight-foward, given what we have already shown.

$$\vdash \quad \langle \mathsf{N}_\mathsf{u} \rangle \phi \leftrightarrow (\mathsf{pre}(\mathsf{u}) \wedge [\mathsf{N}_\mathsf{u}]\phi) \tag{42}$$
$$\vdash \quad \langle \mathsf{N}_\mathsf{u} \rangle \phi \leftrightarrow \mathsf{pre}(\mathsf{u}) \tag{43}$$
$$\vdash \quad \langle \mathsf{N}_\mathsf{u} \rangle \phi \leftrightarrow \exists_B \phi \tag{44}$$

(42) follows from a simple semantic argument, (43) follows from (41) above, and (44) follows from the definition of N.

Finally we show that N is a B-restricted action model.

Let $c \in C$, let $\gamma \in \Gamma_c$, let $\mathsf{u} \in \mathsf{S}^\gamma$ and let $M_s \in \mathcal{K}$ be a Kripke model such that $M_s \models \mathsf{pre}^\gamma(\mathsf{u})$. As N^γ is a B-restricted action model, then $M_s \otimes \mathsf{N}^\gamma_\mathsf{u} \leftleftharpoons_B M_s$. As the construction of N includes a complete copy of N^γ, and adds only inward edges to action points in S^γ then it is a simple matter to show that $\mathsf{N}_\mathsf{u} \leftrightharpoons \mathsf{N}^\gamma_\mathsf{u}$. Therefore from Proposition III.5 we have that $M_s \otimes \mathsf{N}_\mathsf{u} \leftrightharpoons M_s \otimes \mathsf{N}^\gamma_\mathsf{u}$. Therefore $M_s \otimes \mathsf{N}_\mathsf{u} \leftleftharpoons_B M_s$.

Let $M_s \in \mathcal{K}$. We note that $M_s \otimes \mathsf{N}_{\mathsf{skip}} \leftrightharpoons M_s$ and therefore $M_s \otimes \mathsf{N}_{\mathsf{skip}} \leftleftharpoons_B M_s$.

Let $M_s \in \mathcal{K}$ such that $M_s \models \mathsf{pre}(\mathsf{u})$. Then for every $a \in A$, $\mathsf{v} \in \mathsf{uR}_a$ and $t \in sR_a$ such that $M_t \models \mathsf{pre}(\mathsf{v})$, from above we have that $M_t \otimes \mathsf{N}_\mathsf{v} \leftleftharpoons_B M_t$ and therefore there exists a B-simulation $\mathfrak{R}^{t,\mathsf{v}}$ between $M_t \otimes \mathsf{N}_\mathsf{v}$ and M_t. Let $M'_{s'} = M_s \otimes \mathsf{N}_\mathsf{v}$. We define the B-simulation $\mathfrak{R}$ between $M'_{s'}$ and M_s as:

$$\mathfrak{R} = \{(s, \mathsf{u})\} \cup \bigcup_{(t,\mathsf{v}) \in s' R'_a} \mathfrak{R}^{t,\mathsf{v}}$$

We must show that $\mathfrak{R}$ satisfies **atoms**, **forth-**a for every $a \in A$, and **back-**a for every $a \in A \setminus B$. We note that if $(u', u) \in \mathfrak{R}^{t,\mathsf{v}}$ for some $(t, \mathsf{v}) \in s' R'_a$ then **atoms**, **forth-**a and **back-**a all follow from the fact that $\mathfrak{R}^{t,\mathsf{v}}$ is a B-simulation. Therefore we need only show that **atoms**, **forth-**a and **back-**a hold for $(s', s) \in \mathfrak{R}$.

atoms: $V'(s') = V'(s, \mathsf{u}) = V(s)$.

forth-a**:** Let $a \in A$ and let $t' \in s' R'_a$. Then from the definition of action model execution, $t' = (t, \mathsf{v})$ for some $t \in sR_a$ and $\mathsf{v} \in \mathsf{uR}_a$. We note that $(t, \mathsf{v}) \in \mathfrak{R}^{t,\mathsf{v}} \subseteq \mathfrak{R}$. Therefore **forth-**$a$ is satisfied.

back-a**:** Let $a \in A \setminus B$ and let $t \in sR_a$.

Suppose that $a \in C$. From (30) above and the definition of the cover operator we have that $\vdash \exists_B \phi \to \Box_a \bigvee_{\gamma \in \Gamma_a} \exists_B \gamma$. As $M_s \models \exists_B \phi$ then $M_s \models \Box_a \bigvee_{\gamma \in \Gamma_a} \exists_B \gamma$, and so $M_t \models \exists_B \gamma$ for some $\gamma \in \Gamma_a$. By hypothesis we have that $\vdash \langle \mathsf{N}^\gamma_{\mathsf{T}^\gamma} \rangle \gamma \leftrightarrow \exists_B \gamma$, and therefore $M_t \models \langle \mathsf{N}^\gamma_{\mathsf{T}^\gamma} \rangle \gamma$. Therefore $M_t \models \langle \mathsf{N}^\gamma_\mathsf{t} \rangle \gamma$ for some $\mathsf{v} \in \mathsf{T}^\gamma$. Therefore $M_t \models \mathsf{pre}(\mathsf{v})$, and so $(t, \mathsf{v}) \in s' R'_a$ and $((t, \mathsf{v}), t) \in \mathfrak{R}^{t,\mathsf{v}} \subseteq \mathfrak{R}$.

Suppose that $a \notin C$. We note that $M_t \models \mathsf{pre}(\mathsf{skip}) = \top$ and so $(t, \mathsf{skip}) \in s' R'_a$ and $((t, \mathsf{skip}), t) \in \mathfrak{R}^{t,\mathsf{skip}} \subseteq \mathfrak{R}$.

Therefore **back-**a holds.

Therefore $M'_{s'} \leftleftharpoons_B M_s$ and so N is a B-restricted action model. ∎

Theorem V.3. *Let $B \subseteq A$ and let $\phi \in \mathcal{L}_{\otimes\forall}$. Then there exists a multi-pointed B-restricted action model $\mathsf{N}_\mathsf{T} \in \mathcal{AM}$ such that $\vdash [\mathsf{N}_\mathsf{T}]\phi$ and $\vdash \langle \mathsf{N}_\mathsf{T} \rangle \phi \leftrightarrow \exists_B \phi$.*

Proof: Without loss of generality, from Lemma IV.5 we may assume that $\phi \in \mathcal{L}$, and from Proposition III.6 we may further assume that ϕ is in cover disjunctive normal form. Then we proceed by induction on the structure of ϕ.

Suppose that $\phi = \pi \wedge \bigwedge_{c \in C} \nabla_c \Gamma_c$ where $\pi \in \mathcal{L}_0$, $C \subseteq A$, and for every $c \in C$: $\Gamma_c \subseteq \mathcal{L}$ is a finite set of formulae. Then from the induction hypothesis for every $c \in C$ and $\gamma \in \Gamma_c$ there exists a multi-pointed B-restricted action model $\mathsf{N}^\gamma_{\mathsf{T}^\gamma} \in \mathcal{AM}$ such that $\vdash [\mathsf{N}^\gamma_{\mathsf{T}^\gamma}]\gamma$ and $\vdash \langle \mathsf{N}^\gamma_{\mathsf{T}^\gamma} \rangle \gamma \leftrightarrow \exists_B \gamma$. Then from Lemma V.2 there exists a multi-pointed B-restricted action model N_T such that $\vdash [\mathsf{N}_\mathsf{T}]\phi$ and $\vdash \langle \mathsf{N}_\mathsf{T} \rangle \phi \leftrightarrow \exists_B \phi$. We note that the base case for the induction occurs when $\phi = \pi$ (i.e.

$C = \emptyset$) or when $\phi = \pi \wedge \bigwedge_{c \in C} \nabla_c \emptyset$ (i.e. for every $c \in C$: $\Gamma_c = \emptyset$).

Suppose that $\phi = \alpha \vee \beta$. Then from the induction hypothesis there exists multi-pointed B-restricted action models $\mathsf{N}^{\alpha}_{\mathsf{T}^\alpha}, \mathsf{N}^{\beta}_{\mathsf{T}^\beta} \in \mathcal{AM}$ such that for $\gamma \in \{\alpha, \beta\}$: $[\mathsf{N}^{\gamma}_{\mathsf{T}^\gamma}]\gamma$ and $\langle \mathsf{N}^{\gamma}_{\mathsf{T}^\gamma} \rangle \gamma \leftrightarrow \exists_B \gamma$. Then from Lemma V.1 there exists a multi-pointed B-restricted action model N_T such that $[\mathsf{N}_\mathsf{T}]\phi$ and $\langle \mathsf{N}_\mathsf{T} \rangle \phi \leftrightarrow \exists_B \phi$. ∎

This result allows us to show a complete correspondence between action model quantifiers and refinement quantifiers.

Corollary V.4. *The semantics of arbitrary action model logic of Definition III.6 and Definition III.8 are equivalent for the logic of $K_{\otimes\forall}$.*

Proof: For convenience we will show that the dual statement of the semantics of Definition III.6 and Definition III.8 are equivalent. The dual statement of Definition III.6 is:

$$M_s \vDash \exists_B \phi \quad \text{iff} \quad \text{there exists } M'_{s'} \in \mathcal{K} \text{ such that } M'_{s'} \preceq_B M_s \text{ and } M'_{s'} \vDash \phi$$

and the dual statement of Definition III.8 is:

$$M_s \vDash \exists_B \phi \quad \text{iff} \quad \text{there exists } \mathsf{N}_\mathsf{u} \in \mathcal{AM}_B \text{ such that } M_s \vDash \langle \mathsf{N}_\mathsf{u} \rangle \phi$$

Let $\phi \in \mathcal{L}_{\otimes\forall}$ and let $M_s \in \mathcal{K}$.

Suppose that there exists $M'_{s'} \in \mathcal{K}$ such that $M'_{s'} \preceq_B M_s$ and $M'_{s'} \vDash \phi$. Then $M_s \vDash \exists_B \phi$. From Theorem V.3 there exists a multi-pointed B-restricted action model N_T such that $\langle \mathsf{N}_\mathsf{T} \rangle \phi \leftrightarrow \exists_B \phi$. Therefore $M_s \vDash \langle \mathsf{N}_\mathsf{T} \rangle \phi$ and in particular $M_s \vDash \langle \mathsf{N}_\mathsf{u} \rangle \phi$ for some $\mathsf{u} \in \mathsf{T}$. Therefore there exists some $\mathsf{N}_\mathsf{u} \in \mathcal{AM}_B$ such that $M_s \vDash \exists \mathsf{N}_\mathsf{u} \phi$.

Suppose that there exists some $\mathsf{N}_\mathsf{u} \in \mathcal{AM}_B$ such that $M_s \vDash \exists \mathsf{N}_\mathsf{u} \phi$. Let $M'_{s'} = M_s \otimes \mathsf{N}_\mathsf{u}$. Then $M'_{s'} \vDash \phi$, and as N_u is a B-restricted action model, by definition $M'_{s'} \preceq_B M_s$. ∎

Corollary V.5. *Let $B \subseteq A$ and let $\phi \in \mathcal{L}$ be a formula in cover disjunctive normal form of size n. Then there exists a multi-pointed B-restricted action model $\mathsf{N}_\mathsf{T} \in \mathcal{AM}_B$ such that $[\mathsf{N}_\mathsf{T}]\phi$ and $\langle \mathsf{N}_\mathsf{T} \rangle \phi \leftrightarrow \exists_B \phi$ such that N_T is of size $O(n^2)$.*

We note that conversion from $\mathcal{L}_{\otimes\forall}$ to $\mathcal{L}$ may result in a considerable increase in the size of the formula; Bozzelli, van Ditmarsch and Pinchinat [8] gave an exponential lower-bound on the succinctness of the refinement modal logic that also applies to the arbitrary action model logic. Therefore although the size of our synthesised action model is $O(n^2)$ with respect to the size of a formula in cover disjunctive normal form, it may be considerably larger for arbitrary $\mathcal{L}_{\otimes\forall}$ formulae.

VI. Future Work

We have yet to consider the arbitrary action model logic in the setting of other classes of Kripke models, such as $\mathcal{KD}45$ or $\mathcal{S}5$, the addition of common knowledge operators to the language, or the question of succinctness or complexity results.

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