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Refinement Quantified Logics of Knowledge

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Abstract

Refinement quantifiers were introduced to quantify over all refinements of a model in modal logic, where a refinement is described by a simulation relation. Given the "possible worlds" interpretation of modal logic, a refinement corresponds to an agent ruling out certain possible worlds based on new information. Recent work by van Ditmarsch, French and Pinchinat has presented an axiomatization and complexity results for refinement quantifiers in the general modal logic **K**. Here we extend these results to apply to the epistemic and doxastic settings for a single agent.

Keywords: Epistemic Logic, Refinement Quantification, Axiomatizations, Decision Procedures.

1 Introduction

Epistemic logic is a modal logic used to reason about the knowledge that a collection of agents hold about the state of the world. Dynamic epistemic logic considers how this knowledge may change in response to informative updates, such as announcements or messages, which introduce new information to some of the agents.

Previous work in dynamic epistemic logic has considered extensions of epistemic logic which allow informative updates to be modelled and reasoned about. Of particular interest to us are logics which introduce operators for reasoning about the results of a specific informative update, or operators for quantifying over arbitrary informative updates.

Several representations of informative updates have been considered for such logics. Public announcements are a simple, restrictive representation of informative update; a public announcement is simply an epistemic formula which is publicly announced to all agents in the system. Action models are more complex representations of informative updates, capable of representing informative updates in which

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information is given privately to only a subset of agents in the system, something which is not possible with a public announcement. Action models are capable of expressing any public announcement.

Public announcement logic, introduced by Plaza [10], and also independently by Gerbrandy and Groenvald [8], extends epistemic logic with an operator for reasoning about the result of a specific public announcement. A similar logic for action models was introduced by Baltag and Moss [3].

Balbiani et al. [2] then explored arbitrary public announcement logic, an extension of public announcement logic which provides an operator for quantifying over arbitrary public announcements. van Ditmarsch and French [5] later proved that arbitrary public announcement logic was undecidable in the multi-agent setting, and so similar logics for action models may also be undecidable.

van Ditmarsch and French then went on to introduce future event logic [6], an extension of modal logic which introduces an operator for quantifying over refinements of models. The finite refinements of a model are equivalent to the models which result from the execution of arbitrary action models. van Ditmarsch, French, and Pinchinat [7] later showed that future event logic is decidable with the underlying modal axioms of $\mathbf{K}_{(1)}$.

The present work extends future event logic to the setting for which it was originally intended: epistemic logic. We examine both epistemic logic and doxastic logic, provide an axiomatisation of the single-agent form of both logics, and an algorithm providing an upper-bound for the complexity of satisfiability in both logics.

2 Technical preliminaries

We recall the definitions given by van Ditmarsch, French, and Pinchinat [7] in describing the future event logic, and adapt those definitions to be based on epistemic logic, $\mathcal{L}^{\mathbf{S5}}$, and doxastic logic, $\mathcal{L}^{\mathbf{KD45}}$. Specifically, we restrict the Kripke models under discussion to those in the class of *S5* models when we are discussing the extension of the future event logic to $\mathcal{L}^{\mathbf{S5}}$, and to those in the class of *KD45* models when we are discussing the extension to $\mathcal{L}^{\mathbf{KD45}}$.

Let A be a non-empty, finite set of agents, and let P be a non-empty, countable set of propositional atoms.

Definition 2.1 [Kripke model] A *Kripke model* $M = (S, R, V)$ consists of a *domain* S, which is a set of states (or worlds), *accessibility* $R : A \to \mathcal{P}(S \times S)$, and a *valuation* $V : P \to \mathcal{P}(S)$. The class of all Kripke models is called K. We write $M \in K$ to denote that M is a Kripke model.

For $R(a)$ we write R_a. Given two states $s, s' \in S$, we write $R_a(s, s')$ to denote that $(s, s') \in R_a$. We write sR_a for $\{t|(s, t) \in R_a\}$. As we will be required to discuss several models at once, we will use the convention that $M = (S^M, R^M, V^M)$, $N = (S^N, R^N, V^N)$, etc. For $s \in S^M$ we will let M_s refer to the pair (M, s), or the pointed Kripke model M at state s.

Definition 2.2 [Epistemic model] An *epistemic model* is a Kripke model $M = (S, R, V)$ such that the relation R_a is an equivalence relation for all $a \in A$. The class of all epistemic models is called *S5*. We write $M \in S5$ to denote that M is an epistemic model.

Definition 2.3 [Doxastic model] A *doxastic model* is a Kripke model $M = (S, R, V)$ such that the relation R_a is serial, transitive, and Euclidean for all $a \in A$. The class of all doxastic models is called *KD45*. We write $M \in KD45$ to denote that M is a doxastic model.

Throughout this paper we present results in both *S5* and *KD45*, often showing a result for *S5* first and then the same result in *KD45*. As such we will assume that all models are epistemic models when discussing $\mathcal{L}^{\mathbf{S5}}$ or $\mathcal{L}^{\mathbf{S5}}_{\triangleright}$, and that all models are doxastic models when discussing $\mathcal{L}^{\mathbf{KD45}}$ or $\mathcal{L}^{\mathbf{KD45}}_{\triangleright}$.

Definition 2.4 [Bisimulation] Let $M = (S, R, V)$ and $M' = (S', R', V')$ be Kripke models. A non-empty relation $\mathcal{R} \subseteq S \times S'$ is a *bisimulation* if and only if for all $s \in S$ and $s' \in S'$, with $(s, s') \in \mathcal{R}$, for all $a \in A$:

atoms $s \in V(p)$ if and only if $s' \in V'(p)$ for all $p \in P$

forth-a for all $t \in S$, if $R_a(s, t)$, then there is a $t' \in S'$ such that $R'_a(s', t')$ and $(t, t') \in \mathcal{R}$

back-a for all $t' \in S'$, if $R'_a(s', t')$, then there is a $t \in S$ such that $R_a(s, t)$ and $(t, t') \in \mathcal{R}$.

We call M_s and $M'_{s'}$ bisimilar, and write $M_s \leftrightarrows M'_{s'}$ to denote that there is a bisimulation between M_s and $M'_{s'}$.

Definition 2.5 [Simulation and refinement] Let M and M' be Kripke models. A non-empty relation $\mathcal{R} \subseteq S \times S'$ is a *simulation* if and only if it satisfies **atoms** and **forth-**a for every $a \in A$.

We call $M'_{s'}$ a simulation of M_s and we call M_s a refinement of $M'_{s'}$. We write $M'_{s'} \succeq M_s$ to denote this, or alternatively, $M_s \preceq M'_{s'}$.

A relation that satisfies **atoms** and **forth-**b for every $b \in A$, and satisfies **back-**b for every $b \in A - \{a\}$, for some $a \in A$, is an *a-simulation.*

We call $M'_{s'}$ an a-simulation of M_s, and we call M_s an a-refinement of $M'_{s'}$. We write $M'_{s'} \succeq_a M_s$ to denote this, or alternatively, $M_s \preceq_a M'_{s'}$.

We will use a-refinements to define the semantics of the future event logic. The significance of refinements is that the refinements of a finite Kripke model are exactly the models that result from the execution of an arbitrary action model [6].

Finally we give a result that will be used in the complexity results later.

Lemma 2.6 *The relation $\succeq_a$ is reflexive, transitive and satisfies the Church-Rosser property in the class of models K.*

This is shown by van Ditmarsch, French and Pinchinat [7].

3 Syntax and semantics

Here we define the syntax and semantics of the logics $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$, which restrict the logic $\mathcal{L}_{\triangleright}^{\mathbf{K}}$, defined by van Ditmarsch, French, and Pinchinat to models and refinements of models that are in *S5* or *KD45* respectively.

The same syntax used for $\mathcal{L}_{\triangleright}^{\mathbf{K}}$ is used for $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$, and so we will define it only once, as $\mathcal{L}_{\triangleright}$. We also refer to the language of modal formulae as $\mathcal{L}$, which is $\mathcal{L}_{\triangleright}$ without the $\blacktriangleright_a$ operator.

Definition 3.1 [Language of $\mathcal{L}_{\triangleright}$] Given a finite set of agents A and a set of propositional atoms P, the language of $\mathcal{L}_{\triangleright}$ is inductively defined as

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a\phi \mid \blacktriangleright_a\phi$$

where $a \in A$ and $p \in P$.

Standard abbreviations include: $\bot ::= p \wedge \neg p$; $\top ::= \neg\bot$; $\phi \vee \psi ::= \neg(\neg\phi \wedge \neg\psi)$; $\phi \rightarrow \psi ::= \neg\phi \vee \psi$; and $\Diamond_a\phi ::= \neg\Box_a\neg\phi$. We use the abbreviation $\triangleright_a\phi ::= \neg\blacktriangleright_a\neg\phi$ for the dual of the $\blacktriangleright_a$ operator.

We also use the cover operator $\nabla_a\Gamma$, which is an abbreviation for $\Box_a \bigvee_{\gamma\in\Gamma} \gamma \wedge \bigwedge_{\gamma\in\Gamma} \Diamond_a\gamma$, where Γ is a finite set of formulae. This is the basis of our axiomatisation, as it is for the axiomatisation of $\mathcal{L}_{\triangleright(1)}^{\mathbf{K}}$ presented by van Ditmarsch, French and Pinchinat [7].

The semantics for $\mathcal{L}_{\triangleright}^{\mathbf{K}}$, $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$ are very similar, and so we will introduce a generalised set of semantics that can be applied to all three.

Definition 3.2 [Semantics of $\mathcal{L}_{\triangleright}^{\mathbf{C}}$] Let C be a class of Kripke models, and let $M = (S, R, V)$ be a Kripke model taken from the class C. The interpretation of $\phi \in \mathcal{L}_{\triangleright}^{\mathbf{C}}$ is defined by induction.

$$M_s \models p \text{ iff } s \in V_p$$

$$M_s \models \neg\phi \text{ iff } M_s \not\models \phi$$

$$M_s \models \phi \wedge \psi \text{ iff } M_s \models \phi \text{ and } M_s \models \psi$$

$$M_s \models \Box_a\phi \text{ iff for all } t \in S : (s,t) \in R_a \text{ implies } M_t \models \phi$$

$$M_s \models \blacktriangleright_a\phi \text{ iff for all } M'_{s'} \in C : M_s \succeq_a M'_{s'} \text{ implies } M'_{s'} \models \phi$$

The logics $\mathcal{L}_{\triangleright}^{\mathbf{K}}$, $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$ are instances of $\mathcal{L}_{\triangleright}^{\mathbf{C}}$. The difference between these logics are the class of models that formulae are interpreted over. It should be emphasised that the interpretation of the refinement operator, $\blacktriangleright_a$, varies for each logic, as the refinements considered in the interpretation of each logic must be taken from the appropriate class of Kripke models. It is for this reason that $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$ are not conservative extensions of $\mathcal{L}_{\triangleright}^{\mathbf{K}}$. For example, $\triangleright_a\Box_a\bot$ is valid in $\mathcal{L}_{\triangleright}^{\mathbf{K}}$, but not in $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ or $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$. This is because given any pointed model in K, one can construct an a-refinement from that model by deleting the a-edges starting at the designated state; in this resulting a-refinement, $\Box_a\bot$ is satisfied, and hence

$\triangleright_a \Box_a \bot$ is satisfied in the original model. However because of the serial property of *S5* and *KD45* models, $\Box_a \bot$ is not even satisfiable in $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ or $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$, and hence $\triangleright_a \Box_a \bot$ is not satisfiable as well.

Lemma 3.3 *The logics $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$ are bisimulation invariant.*

The proof for bisimulation invariance in $\mathcal{L}_{\triangleright}^{\mathbf{K}}$, given by van Ditmarsch, French and Pinchinat [7] applies to $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$.

Following are some examples illustrating the semantics of $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$.

Example 3.4 Consider a situation where two agents are initially uncertain about a proposition p, where in fact p is true, represented by Figure 1. We assume that all models are in *S5*. An informative event is possible after which a knows that p is true, but b does not know this. This is expressed by:

$$\triangleright_a (\Box_a p \land \neg \Box_b \Box_a p)$$

Example 3.5 Imagine a scenario where an agent is presented with three cards face down, and asked to identify which is the ace of spades (let's suppose it is the $left$ card). An agent may receive many informative updates (e.g. "the card with the bent corner is definitely not the ace"), but as an agent's knowledge is only ever based on reliable evidence, it follows that given any informative update, there is always a further informative update after which the agent knows the location of the ace:

$$left \rightarrow \blacktriangleright \triangleright \Box left. \tag{1}$$

This scenario is represented in Figure 2. We can also imagine a corresponding scenario in terms of the agent's *belief* rather than *knowledge*. That is we suppose the axioms of $\mathcal{L}^{\mathbf{KD45}}$ rather than $\mathcal{L}^{\mathbf{S5}}$. Here an agent may believe that the ace is either the *centre* card or the *right* card, despite this not being the case. Again the agent may receive informative updates, but the formula (1) does not hold. We also note that once an agent holds a belief, no informative update will cause the agent to revise that belief:

$$\Box(right \lor centre) \rightarrow \blacktriangleright \Box(right \lor centre). \tag{2}$$

That is, we do not consider belief *revision* in the sense of [1] but rather belief *refinement*. This situation is depicted in Figure 3. This allows incorrect information,

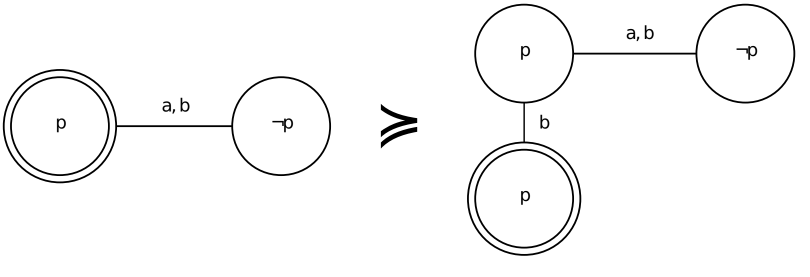

Fig. 1. The initial state where a and b are uncertain about p, with the subsequent refinement

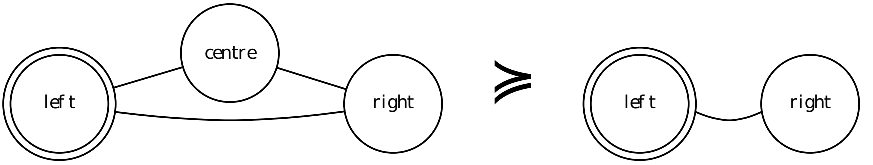

Fig. 2. The initial state of an agent's uncertainty, with an example refinement, both in *S5*.

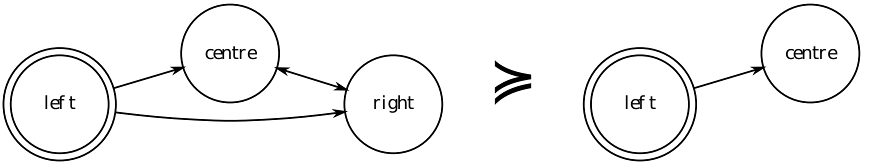

Fig. 3. The initial setup in K, where an agent believes the ace is either the centre or right card, with an example refinement. Note that the relations are not equivalences, and so directions of relations are marked, and we do not have reflexivity.

but requires that the information provided is consistent because of the requirement that *KD45* models are serial.

The axiomatisation given in this paper is only for the single-agent variant of $\mathcal{L}_{\triangleright}^{\mathbf{S5}}$ and $\mathcal{L}_{\triangleright}^{\mathbf{KD45}}$. The differences for the multi-agent variants will be mentioned briefly in the future work section. We note that in the single agent case for *S5*, the semantic interpretation of the refinement quantifier $\blacktriangleright$ agrees with the semantic interpretation of the arbitrary public announcement operator of Balbiani, et al. [2]. This is because in the single agent case of *S5*, the refinements of a model, up to bisimulation, are formed by a restriction on the set of states of the original model; a public announcement is similarly formed by such a restriction. However, the resulting axiomatizations are quite different since the axiomatization for arbitrary public announcement operators relies heavily on public announcements, whereas we do not require such operators.

When discussing the single-agent variants of these logics, we will use a subscript (1) to denote the single-agent logics, e.g. $\mathcal{L}_{(1)}^{\mathbf{S5}}$, $\mathcal{L}_{\triangleright(1)}^{\mathbf{S5}}$, and so on. We also drop the superfluous subscripts denoting agents in the syntax of the logics, e.g. we write $\Box$ instead of $\Box_a$.

To simplify the completeness proof of our axiomatisations, we rely on the fact that all formulae of $\mathcal{L}$ are equivalent to formulae in cover logic prenex normal form, under both the semantics of $\mathcal{L}_{(1)}^{\mathbf{S5}}$ and $\mathcal{L}_{(1)}^{\mathbf{KD45}}$. We define the prenex normal form first, then give another definition for the form using the cover operator, and then show our equivalence result.

Definition 3.6 [Prenex normal form] A formula in prenex normal form is specified by the following abstract syntax:

$$\alpha ::= \delta \mid \alpha \vee \alpha$$
$$\delta ::= \pi \mid \Box\pi \mid \Diamond\pi \mid \delta \wedge \delta$$

where π stands for a propositional formula.

Lemma 3.7 *Every formula in $\mathcal{L}$ is equivalent to a formula in prenex normal form,* *under the semantics of* $\mathcal{L}^{\mathbf{S5}}_{(1)}$.

This is shown by Meyer and van der Hoek [9].

Lemma 3.8 *Every formula in* $\mathcal{L}$ *is equivalent to a formula in prenex normal form, under the semantics of* $\mathcal{L}^{\mathbf{KD45}}_{(1)}$.

Proof. The proof given by Meyer and van der Hoek [9] for Lemma 3.7 applies also to $\mathcal{L}^{\mathbf{KD45}}_{(1)}$.

Meyer and van der Hoek remarked that the only use of the reflexivity axiom of $\mathcal{L}^{\mathbf{S5}}$, $\mathbf{T}$, in the proof, is in the form of the theorems $\vdash \Box\Box\phi \to \Box\phi$, and $\vdash \Box\neg\Box\phi \to \neg\Box\phi$. Therefore the proof holds for any logic which replaces $\mathbf{T}$ with axioms entailing both of these properties. Both of these properties are obviously valid in $\mathcal{L}^{\mathbf{KD45}}_{(1)}$, and therefore the proof of Lemma 3.7 applies to this result. □

Definition 3.9 [Cover logic prenex normal form] A formula in cover logic prenex normal form is specified by the following abstract syntax:

$$\alpha ::= \pi \wedge \nabla\Gamma \mid \alpha \vee \alpha$$

where π is a propositional formula, and Γ is a set of propositional formulae.

Lemma 3.10 *Every formula in* $\mathcal{L}$ *is equivalent to a formula in cover logic prenex normal form, under both the semantics of* $\mathcal{L}^{\mathbf{S5}}_{(1)}$ *and* $\mathcal{L}^{\mathbf{KD45}}_{(1)}$.

Proof. Without loss of generality, we may assume that our given formula is in prenex normal form (by Lemma 3.7 for $\mathcal{L}^{\mathbf{S5}}_{(1)}$, or by Lemma 3.8 for $\mathcal{L}^{\mathbf{KD45}}_{(1)}$).

Given a formula in prenex normal form, we consider each disjunct separately. We can convert each term $\Box\gamma$ or $\Diamond\gamma$ into an equivalent term using the cover operator, using the equivalences $\Box\gamma \equiv \nabla\{\gamma\}$ and $\Diamond\gamma \equiv \nabla\{\gamma, \top\}$ Note that each resulting term contains a cover operator applied only to a set of propositional formulae.

An inductive argument can be used to show that we can collapse the resulting conjunction of cover operators into a single term containing one cover operator applied to a set of propositional formulae. We use the following equivalence to achieve this, and note that at each stage this equivalence preserves the property that the cover operator is only applied to a set of propositional formulae.

$$\nabla\Gamma \wedge \nabla\Gamma' \equiv \nabla\big(\{\gamma \wedge \bigvee_{\gamma' \in \Gamma'} \gamma' \mid \gamma \in \Gamma\} \cup \{\gamma' \wedge \bigvee_{\gamma \in \Gamma} \gamma \mid \gamma' \in \Gamma'\}\big)$$

Repeating this for each disjunct in our original formula leaves us with a formula in cover logic prenex normal form. □

The cover logic prenex normal form will be used in our completeness proofs.

4 Axiomatisation

Here we present sound and complete axiomatisations of $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$. As they share common rules and axioms, we will define the axiomatisation $\mathbf{FEL}_{(1)}$ containing these rules and axioms, and then define the axiomatisations of $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ as extensions of $\mathbf{FEL}_{(1)}$.

Definition 4.1 The axiomatisation $\mathbf{FEL}_{(1)}$ is a substitution schema of the following axioms:

P All tautologies of propositional logic.

K $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$

G0 $\blacktriangleright(\phi \to \psi) \to \blacktriangleright\phi \to \blacktriangleright\psi$

G1 $\blacktriangleright\alpha \leftrightarrow \alpha$, where α is a propositional formula.

Along with the rules:

MP From $\vdash \phi \to \psi$ and $\vdash \phi$ infer $\vdash \psi$.

Nec1 From $\vdash \phi$ infer $\vdash \Box\phi$.

Nec2 From $\vdash \phi$ infer $\vdash \blacktriangleright\phi$.

Definition 4.2 The axiomatisation $\mathbf{FEL}^{\mathbf{K}}_{(1)}$ is a substitution schema consisting of the axioms and rules of $\mathbf{FEL}_{(1)}$, along with the additional axiom:

GK $\triangleright\nabla\Gamma \leftrightarrow \bigwedge_{\gamma\in\Gamma} \Diamond\triangleright\gamma$

Definition 4.3 The axiomatisation $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ is a substitution schema consisting of the axioms and rules of $\mathbf{FEL}_{(1)}$, along with the additional axioms:

T $\Box\phi \to \phi$

5 $\Diamond\phi \to \Box\Diamond\phi$

GS5 $\triangleright\nabla\Gamma \leftrightarrow \bigvee_{\gamma\in\Gamma} \gamma \wedge \bigwedge_{\gamma\in\Gamma} \Diamond\gamma$, where Γ is a set of propositional formulae.

Definition 4.4 The axiomatisation $\mathbf{FEL}^{\mathbf{KD45}}_{(1)}$ is a substitution schema consisting of the axioms and rules of $\mathbf{FEL}_{(1)}$, along with the additional axioms:

D $\Box\phi \to \Diamond\phi$

4 $\Box\phi \to \Box\Box\phi$

5 $\Diamond\phi \to \Box\Diamond\phi$

GKD45 $\triangleright\nabla\Gamma \leftrightarrow \bigwedge_{\gamma\in\Gamma} \Diamond\gamma$, where Γ is a set of propositional formulae.

We note that many of the axioms from $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ are also axioms for the logics $\mathcal{L}^{\mathbf{S5}}_{(1)}$, $\mathcal{L}^{\mathbf{KD45}}_{(1)}$ and $\mathcal{L}^{\mathbf{K}}_{\triangleright(1)}$.

Lemma 4.5 *The axioms* $\mathbf{FEL}_{(1)}$ *are sound for the logics* $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ *and* $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$.

Proof. The soundness of the axioms **P** and **K**, and the rules **MP** and **Nec1** in $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ can be shown by the same reasoning that they are sound in $\mathcal{L}^{\mathbf{S5}}_{(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{(1)}$. The soundness of the axioms **G0** and **G1**, and the rule **Nec2** can be shown by the same reasoning used by van Ditmarsch, French, and Pinchinat [7] to prove their soundness in $\mathcal{L}^{\mathbf{K}}_{\triangleright(1)}$. □

Lemma 4.6 *The axiomatisation* $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ *is sound for the logic* $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$.

Proof. Soundness of the axioms **P**, **K**, **G0**, and **G1**, and the rules **MP**, **Nec1** and **Nec2** are shown above. Soundness of the axioms **T** and **5** in $\mathcal{L}^{\mathbf{S5}}_{(1)}$ are well-known results, and their soundness in $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ follows from this. All that is to be shown is the soundness of **GS5**.

Let Γ be a finite set of propositional formulae, and let M_s be a model in *S5* such that $M_s \models \bigvee_{\gamma\in\Gamma}\gamma \wedge \bigwedge_{\gamma\in\Gamma}\Diamond\gamma$.

Then for each $\gamma \in \Gamma$, there is some successor $t^\gamma \in sR^M$ such that $M_{t^\gamma} \models \gamma$.

We can construct the model M' such that $S^{M'} = \{s\} \cup \{t^\gamma \mid \gamma \in \Gamma\}$, $R^{M'} = S^{M'} \times S^{M'}$ and for all $p \in P$, $V^{M'}(p) = V^M(p) \cap S^{M'}$. This model is clearly in *S5*.

By construction, for each $\gamma \in \Gamma$, there is a successor $t^\gamma \in sR^{M'}$ such that $M'_{t^\gamma} \models \gamma$. For each successor $t \in sR^{M'}$ we have that $M'_t \models \bigvee_{\gamma\in\Gamma}\gamma$, as each successor is either one of the t^γ for some $\gamma \in \Gamma$, in which case $M'_{t^\gamma} \models \gamma$, or it is our initial state s, in which case $M'_s \models \bigvee_{\gamma\in\Gamma}\gamma$ follows from our hypothesis that $M_s \models \bigvee_{\gamma\in\Gamma}\gamma$. Therefore $M'_s \models \nabla\Gamma$.

Furthermore we have that $M'_s \preceq M_s$ by the relation $\mathcal{R} = \{(s,s)\} \cup \{(t^\gamma, t^\gamma) \mid \gamma \in \Gamma\}$ (**atoms** and **forth** are satisfied). Therefore $M_s \models \triangleright\nabla\Gamma$.

Conversely, let Γ be a finite set of propositional formulae, and let M_s be a model in *S5* such that $M_s \models \triangleright\nabla\Gamma$. Then there exists some model $M'_{s'} \preceq M_s$ in *S5*, via some simulation $\mathcal{R} \subseteq S' \times S$, such that $M'_{s'} \models \nabla\Gamma$.

From the definition of the cover operator, $M'_{s'} \models \Box\bigvee_{\gamma\in\Gamma}\gamma \wedge \bigwedge_{\gamma\in\Gamma}\Diamond\gamma$.

As M' is in *S5*, we know that $s' \in s'R^{M'}$, and so it follows from $M'_{s'} \models \Box\bigvee_{\gamma\in\Gamma}\gamma$ that $M'_{s'} \models \bigvee_{\gamma\in\Gamma}\gamma$. As we know that $(s', s) \in \mathcal{R}$, from **atoms** we know that M_s and $M'_{s'}$ are equivalent for propositional formulae. As each $\gamma \in \Gamma$ is propositional, it follows that $M_s \models \bigvee_{\gamma\in\Gamma}\gamma$.

Furthermore, from $M'_{s'} \models \bigwedge_{\gamma\in\Gamma}\Diamond\gamma$, we know that for every $\gamma \in \Gamma$, there exists some $t'_\gamma \in s'R^{M'}$ such that $M'_{t'_\gamma} \models \gamma$. It then follows from **forth** that there exists some $t_\gamma \in S^M$ such that $t_\gamma \in sR^M$ and $(t'_\gamma, t_\gamma) \in \mathcal{R}$. From **atoms** we know that M_{t_γ} and $M'_{t'_\gamma}$ are equivalent for propositional formulae. As γ is propositional, it follows that $M_{t_\gamma} \models \gamma$ and therefore $M_s \models \Diamond\gamma$ for each $\gamma \in \Gamma$. Therefore $M_s \models \bigwedge_{\gamma\in\Gamma}\Diamond\gamma$, and so $M_s \models \bigvee_{\gamma\in\Gamma}\gamma \wedge \bigwedge_{\gamma\in\Gamma}\Diamond\gamma$.

Therefore **GS5** is sound, and so $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ is sound for the logic $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$. □

Lemma 4.7 *The axiomatisation* $\mathbf{FEL}^{\mathbf{KD45}}_{(1)}$ *is sound for the logic* $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$.

Proof. The proof is similar to the proof for Lemma 4.6. Instead of showing soundness for the $\mathbf{S5}_{(1)}$ axioms, we must show that the $\mathbf{KD45}_{(1)}$ axioms are sound, and this follows from their soundness in $\mathcal{L}^{\mathbf{KD45}}_{(1)}$. The main difference in the proof of soundness is for **GKD45** as compared to the proof for **GS5**, is that in the right to left direction, we do not have to show that $M_s \vDash \bigvee_{\gamma\in\Gamma} \gamma$; as doxastic models do not have to be reflexive, there is no requirement for s to be in the possible worlds for the constructed refinement. For the left to right direction of the proof, the refinement $M'_{s'}$ is a KD45 model instead of an S5 model, but this has no bearing on the rest of the proof. □

We show the completeness of the axiomatisations $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ and $\mathbf{FEL}^{\mathbf{KD45}}_{(1)}$ by provably correct translations from $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ to $\mathcal{L}^{\mathbf{S5}}_{(1)}$, and from $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ to $\mathcal{L}^{\mathbf{KD45}}_{(1)}$ respectively. Completeness then follows from the completeness of $\mathcal{L}^{\mathbf{S5}}_{(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{(1)}$.

Lemma 4.8 *Every formula of $\mathcal{L}_{\triangleright(1)}$ is provably equivalent to a formula of $\mathcal{L}_{(1)}$ with the axiomatisation $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$.*

Proof. Given a formula ψ we prove by induction on the number of occurrences of $\triangleright$ that ψ is equivalent to a $\triangleright$-free formula, and therefore to a formula in $\mathcal{L}_{(1)}$. The base case with no $\triangleright$ operators is trivial, as a $\triangleright$-free formula is a formula in $\mathcal{L}$. Now assume that ψ contains $n+1$ $\triangleright$-operators. Choose a subformula of type $\triangleright\phi$ of our given formula, where ϕ is $\triangleright$-free. Without loss of generality, by Lemma 3.10 we can assume that ϕ is in cover logic prenex normal form. We prove by induction on the structure of ϕ that $\triangleright\phi$ is provably equivalent to a formula χ without the $\triangleright$ operator.

- $\triangleright(\phi \vee \psi)$ iff $\triangleright\phi \vee \triangleright\psi$. (Derivable from **P**, **G0**, and **G1**, and the induction hypothesis.)
- $\triangleright(\pi \wedge \nabla\Gamma)$ iff $\pi \wedge \bigvee_{\gamma\in\Gamma} \gamma \wedge \bigwedge_{\gamma\in\Gamma} \nabla\{\gamma, \top\}$. (Derivable from **P**, **G0**, **G1**, and **GS5**, and the induction hypothesis)

Replacing $\triangleright\phi$ with χ in ψ gives an equivalent formula with one less $\triangleright$-operator. Thus by induction, all formulae in $\mathcal{L}_{\triangleright(1)}$ can be translated into an equivalent formula in $\mathcal{L}_{(1)}$ using the axiomatisation $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$. □

Lemma 4.9 *Every formula of $\mathcal{L}_{\triangleright(1)}$ is provably equivalent to a formula of $\mathcal{L}_{(1)}$ with the axiomatisation $\mathbf{FEL}^{\mathbf{KD45}}_{(1)}$.*

Proof. The proof is similar to the proof for Lemma 4.8, with the main difference being that the axiom **GKD45** is used in place of **GS5** in the induction over the structure of ϕ.

Specifically, given some subformula $\triangleright\phi$, where ϕ is $\triangleright$-free, we can prove by induction on the structure of ϕ that $\triangleright\phi$ is provably equivalent to a formula χ without $\triangleright\phi$.

- $\triangleright(\phi \vee \psi)$ iff $\triangleright\phi \vee \triangleright\psi$. (Follows from the soundness of **P**, **G0**, and **G1**, and the induction hypothesis.)
- $\triangleright(\pi \wedge \nabla\Gamma)$ iff $\pi \wedge \bigwedge_{\gamma\in\Gamma} \nabla\{\gamma, \top\}$. (Follows from the soundness of **P**, **G0**, **G1**, and **GKD45**, and the induction hypothesis)

The rest is as for Lemma 4.8. □

The rest of the completeness proof is merely a formality to show that, given the above translations into $\mathcal{L}_{(1)}$, we can show completeness by using these translations along with the completeness of $\mathcal{L}^{\mathbf{S5}}_{(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{(1)}$.

Corollary 4.10 *Let $\phi \in \mathcal{L}_{\triangleright(1)}$ be given and $\psi \in \mathcal{L}_{(1)}$ be equivalent to ϕ under the semantics of $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$. If ψ is a theorem in $\mathcal{L}^{\mathbf{S5}}_{(1)}$, then ϕ is a theorem in $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$.*

Proof. Let $\phi \in \mathcal{L}_{\triangleright(1)}$ and let $\psi \in \mathcal{L}_{(1)}$ be semantically equivalent to ϕ. By Lemma 4.8, we can obtain some $\phi' \in \mathcal{L}_{(1)}$ that is semantically equivalent to ϕ (and thus also to ψ) by following the given translation steps. We can extend a derivation of ψ to a derivation of ϕ' as the two are semantically equivalent under $\mathcal{L}^{\mathbf{S5}}_{(1)}$, and by the completeness of $\mathcal{L}^{\mathbf{S5}}_{(1)}$ this equivalence is derivable. As $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ is a conservative extension of $\mathcal{L}^{\mathbf{S5}}_{(1)}$, this equivalence is therefore also derivable in $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$. The derivation can be further extended to ϕ by observing that all of the reduction steps in Lemma 4.8 are provable equivalences in $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$. Therefore ϕ is a theorem in $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$. □

Lemma 4.11 *The axiom schema $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ is complete for the logic $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$.*

In the following proof we use a subscript on the turnstile symbol to denote the logic we are working in (e.g. $\vDash_{\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}}$ is entailment in $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$).

Proof. Let $\phi \in \mathcal{L}_{\triangleright(1)}$ such that $\vDash_{\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}} \phi$. Then by Lemma 4.8, there exists a semantically equivalent formula $\psi \in \mathcal{L}_{(1)}$ which is $\triangleright$-free. As $\vDash_{\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}} \phi$ and $\phi \leftrightarrow \psi$, then $\vDash_{\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}} \psi$. As ψ is $\triangleright$-free, then it follows that $\vDash_{\mathcal{L}^{\mathbf{S5}}_{(1)}} \psi$, and by the completeness of $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ it follows that $\vdash_{\mathcal{L}^{\mathbf{S5}}_{(1)}} \psi$. Therefore by Corollary 4.10 we have that $\vdash_{\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}} \phi$. □

Theorem 4.12 *The axiomatisation $\mathbf{FEL}^{\mathbf{S5}}_{(1)}$ is sound and complete for the logic $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$.*

Proof. The soundness proof is given in Lemma 4.6 and the completeness proof is given in Lemma 4.11. □

Theorem 4.13 *The axiomatisation $\mathbf{FEL}^{\mathbf{KD45}}_{(1)}$ is sound and complete for the logic $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$.*

Proof. The soundness proof is given in Lemma 4.7, and we note that similar results to Corollary 4.10 and Lemma 4.11 can be shown with minor modifications to their proofs, which gives us completeness. □

5 Complexity

The proofs of completeness given above allow us to derive a decision procedure for $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$, and thus show that these logics are decidable.

Theorem 5.1 *The logics $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ are decidable.*

Proof. Given a formula ϕ in $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$, we can find an equivalent ψ in $\mathcal{L}^{\mathbf{S5}}_{(1)}$ (from Lemma 4.8). We can therefore determine whether ψ is satisfiable using a decision procedure designed for $\mathcal{L}^{\mathbf{S5}}_{(1)}$. The decidability for $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ therefore follows from the decidability of $\mathcal{L}^{\mathbf{S5}}_{(1)}$ [4].

The proof for $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ is the same, but relying on Lemma 4.9 for the translation, and on the decidability of $\mathcal{L}^{\mathbf{KD45}}_{(1)}$ [4]. □

In this section we wish to reason about the complexity of the decidability problem for $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$, and specifically we will present an upper-bound for this complexity.

It should be noted that the translation described above has a non-elementary complexity. This comes from the conversion to prenex normal form required for each $\triangleright$-operator. Similar to a disjunctive normal form, this results in an exponential increase in formula size in the worst-case, and this effect is stacked in the case of nested $\triangleright$-operators, resulting in a non-elementary complexity.

Below we describe a better decision procedure for $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$. We first show that to determine the satisfiability of ϕ in $S5_{(1)}$ it is sufficient to only check a finite set of models, $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$, determined by the propositional atoms used in ϕ.

Lemma 5.2 *Let P be a finite set of propositional atoms. Then there exists a finite set $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ of $S5_{(1)}$ models such that, if $\phi \in \mathcal{L}_{\triangleright(1)}$ is a formula over the propositional atoms P then ϕ is satisfiable under the semantics of $\mathcal{L}^{\mathbf{S5}}_{(1)}$ if and only if ϕ is satisfied by a model in $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$*

Proof. Each model in $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ is constructed from a set of states $S \in \mathcal{P}(\mathcal{P}(P)) - \{\emptyset\}$. We note that in $S5_{(1)}$, the relationship between states is uniquely determined from the reflexive, transitive and symmetric properties of $S5_{(1)}$. We define the valuation at each state to be the set of atoms that represents the state itself.

We first note that $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ has the finite model property, as it is expressively equivalent to $\mathcal{L}^{\mathbf{S5}}_{(1)}$, which also has this property [4]. This means that a formula of $\mathcal{L}_{\triangleright(1)}$ is satisfiable in $S5_{(1)}$ if and only if it is satisfied by a finite model in $S5_{(1)}$. Furthermore, we note that every finite model of $S5_{(1)}$ is bisimilar to a finite model in which the valuations at each state are unique. In particular, as $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ is formed from every possible combination of valuations over P, any finite model defined over the propositions in P is bisimilar to a model in $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$.

Then suppose that ϕ is satisfiable. Then it is satisfied by a finite model $M_s \in S5_{(1)}$. Without loss of generality we may assume that the only propositions in M_s are from P. Then M_s is bisimilar to a model in $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ which also satisfies ϕ. Conversely, suppose that ϕ is not satisfiable. Then it is not satisfied by any finite model, and so it is not satisfied by any model in $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$. □

Given the set $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$, we can determine the satisfiability of a formula over the propositional atoms in P by iterating over the models in $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ and testing each model according to the semantics of $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$.

Lemma 5.3 *Algorithm 1 reports that a formula ϕ is satisfiable if and only if ϕ is satisfiable in $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$.*

Proof. $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ is finite and computable, from the description given in the proof of Lemma 5.2. The partial ordering on $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ can be computed due to Lemma 2.6. The correctness of the main loop follows from the semantics of $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ and an inductive argument over the map `satisfies`. □

Corollary 5.4 *Satisfiability for $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ can be determined in 2EXP time.*

Proof. Let ϕ be a formula of size n. There are at most n distinct propositional atoms in ϕ, and so the set of models $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ contains at most $2^{2^n} - 1$ models. If models are represented as sets of states, any two models from $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ can be tested for bisimulation in $O(2^n)$ time, by testing for set inclusion, and so the partial ordering on $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ can be determined in $O(2^n \cdot (2^{2^n} - 1)^2)$ time. There are also at most n subformulae of ϕ, and an ordering of these under inclusion can be computed in $O(n)$ time.

The inner loop of Algorithm 1 executes at most $n \cdot (2^{2^n} - 1) \cdot 2^n$ times. The cases for $\phi \equiv p$ for $p \in P$, $\phi \equiv \neg\alpha$ and $\phi \equiv \alpha \wedge \beta$ can be computed in $O(\lg n)$ time. The case for $\phi \equiv \Box\alpha$ can be computed in $O(2^n)$ time. The case for $\phi \equiv \blacktriangleright\alpha$ can be computed in $O(2^{2^n} - 1)$ time. Therefore the main loop executes in $O(n \cdot (2^{2^n} - 1) \cdot 2^n \cdot (2^{2^n} - 1))$ time, which is $O(2^{O(2^n)})$. □

Algorithm 1. $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$-SAT

Require: Let ϕ be a $\mathcal{L}^{\mathbf{S5}}_{\triangleright(1)}$ formula.
Ensure: Return true if and only if ϕ is satisfiable.
Construct a partial ordering `models` on $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$, ordered by refinement.
Initialise a map `satisfies` from formulae to sets of pointed models. Initialise the values `satisfies`(ψ) $\leftarrow \emptyset$ for every subformula ψ of ϕ.
for all Subformulae ψ of ϕ, in an order that respects the partial ordering of subformula inclusion **do**
  for all $M \in$ `models` **do**
    for all $s \in S^M$ **do**
      Add M_s to `satisfies`(ψ) if and only if any of the following:

$\psi \equiv p$ for some $p \in P$ and $p \in V^M(s)$

$\psi \equiv \neg\alpha$ and $M_s \notin$ `satisfies`(α)

$\psi \equiv \alpha \wedge \beta$ and $M_s \in$ `satisfies`(α) and $M_s \in$ `satisfies`(β)

$\psi \equiv \Box\alpha$ and $M_t \in$ `satisfies`(α) for every $t \in sR^M$

$\psi \equiv \blacktriangleright\alpha$ and $N_t \in$ `satisfies`(α) for every $N_t \in \mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ such that $N_t \preceq M_s$

if `satisfies`(ϕ) $\neq \emptyset$ **then return** true **else return** false

We can define a set similar to $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ for $KD45_{(1)}$ models, called $\mathsf{M}^{\mathbf{KD45}}_{\triangleright(1)}$. Each model in $\mathsf{M}^{\mathbf{KD45}}_{\triangleright(1)}$ is constructed from a set of states $S \in \mathcal{P}(\mathcal{P}(P)) - \{\emptyset\}$, along with an optional state $s \in S$, designated as the only non-reflexive state in the model. We note that in $KD45_{(1)}$, the relationship between states is uniquely determined by which states are reflexive, along with the serial, transitive and Euclidean properties of $KD45_{(1)}$. We can then show a similar result to Lemma 5.2, and extend Algorithm 1 to $\mathcal{L}^{\mathbf{KD45}}_{\triangleright(1)}$ by replacing the set $\mathsf{M}^{\mathbf{S5}}_{\triangleright(1)}$ with $\mathsf{M}^{\mathbf{KD45}}_{\triangleright(1)}$. The resulting algorithm also runs in 2EXP time.

6 Future work

Left for future work are the generalisation of the axiomatisations to the multi-agent logics $\mathcal{L}^{\mathbf{S5}}_{\triangleright}$ and $\mathcal{L}^{\mathbf{KD45}}_{\triangleright}$, and the consideration of the decidability problems in these logics.

The generalisation for the axioms of $\mathcal{L}^{\mathbf{K}}_{\triangleright(1)}$ to the multi-agent logic $\mathcal{L}^{\mathbf{K}}_{\triangleright}$ involves generalising the **GK** axiom, and adding extra axioms to handle the interaction between different agents. Whilst a similar approach does not seem to work for the multi-agent $\mathcal{L}^{\mathbf{S5}}_{\triangleright}$, it seems possible for $\mathcal{L}^{\mathbf{KD45}}_{\triangleright}$.

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